

LINEAR EQUATIONS

Math 21b, O. Knill

SYSTEM OF LINEAR EQUATIONS. A collection of linear equations is called a **system of linear equations**. An example is

$$\begin{cases} 3x - y - z = 0 \\ -x + 2y - z = 0 \\ -x - y + 3z = 9 \end{cases}.$$

This system consists of three equations for three unknowns x, y, z . **Linear** means that no nonlinear terms like $x^2, x^3, xy, yz^3, \sin(x)$ etc. appear. A formal definition of linearity will be given later.

LINEAR EQUATION. The equation $ax+by=c$ is the general linear equation in two variables and $ax+by+cz=d$ is the general linear equation in three variables. The general **linear equation** in n variables has the form $a_1x_1+a_2x_2+\dots+a_nx_n=a_0$. Finitely many of such equations form a **system of linear equations**.

SOLVING BY ELIMINATION.

Eliminate variables. In the first example, the first equation gives $z = 3x - y$. Substituting this into the second and third equation gives

$$\begin{cases} -x + 2y - (3x - y) = 0 \\ -x - y + 3(3x - y) = 9 \end{cases}$$

or

$$\begin{cases} -4x + 3y = 0 \\ 8x - 4y = 9 \end{cases}.$$

The first equation leads to $y = 4/3x$ and plugging this into the other equation gives $8x - 16/3x = 9$ or $8x = 27$ which means $x = 27/8$. The other values $y = 9/2, z = 45/8$ can now be obtained.

SOLVE BY SUITABLE SUBTRACTION.

Addition of equations. If we subtract the third equation from the second, we get $3y - 4z = -9$ and add three times the second equation to the first, we get $5y - 4z = 0$. Subtracting this equation to the previous one gives $-2y = -9$ or $y = 2/9$.

SOLVE BY COMPUTER.

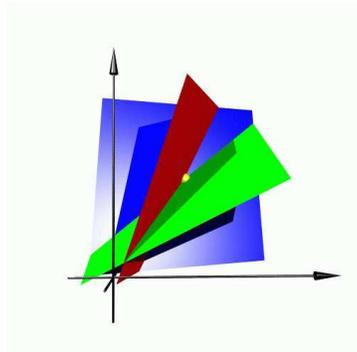
Use the computer. In Mathematica:

$$\text{Solve}\{3x - y - z == 0, -x + 2y - z == 0, -x - y + 3z == 9, \{x, y, z\}\}.$$

But what did Mathematica do to solve this equation? We will look at some algorithms.

GEOMETRIC SOLUTION.

Each of the three equations represents a plane in three-dimensional space. Points on the first plane satisfy the first equation. The second plane is the solution set to the second equation. To satisfy the first two equations means to be on the intersection of these two planes which is here a line. To satisfy all three equations, we have to intersect the line with the plane representing the third equation which is a point.



LINES, PLANES, HYPERPLANES.

The set of points in the plane satisfying $ax + by = c$ form a **line**.

The set of points in space satisfying $ax + by + cz = d$ form a **plane**.

The set of points in n -dimensional space satisfying $a_1x_1 + \dots + a_nx_n = a_0$ define a set called a **hyperplane**.

RIDDLES:

"25 kids have bicycles or tricycles. Together they count 57 wheels. How many have bicycles?"

Solution. With x bicycles and y tricycles, then $x + y = 25, 2x + 3y = 57$. The solution is $x = 18, y = 7$.

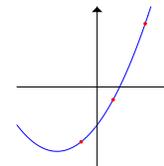
"Tom, the brother of Carry has twice as many sisters as brothers while Carry has equal number of sisters and brothers. How many kids is there in total in this family?"

Solution If there are x brothers and y sisters, then Tom has y sisters and $x - 1$ brothers while Carry has x brothers and $y - 1$ sisters. We know $y = 2(x - 1), x = y - 1$ so that $x + 1 = 2(x - 1)$ and so $x = 3, y = 4$.

INTERPOLATION.

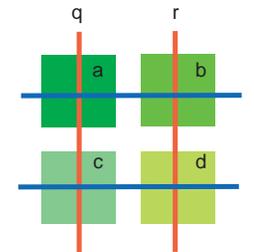
Find the equation of the parabola which passes through the points $P = (0, -1), Q = (1, 4)$ and $R = (2, 13)$.

Solution. Assume the parabola consists of the set of points (x, y) which satisfy the equation $ax^2 + bx + c = y$. So, $c = -1, a + b + c = 4, 4a + 2b + c = 13$. Elimination of c gives $a + b = 5, 4a + 2b = 14$ so that $2b = 6$ and $b = 3, a = 2$. The parabola has the equation $2x^2 + 3x - 1 = 0$



TOMOGRAPHY

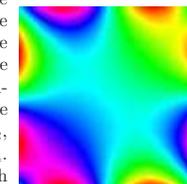
Here is a toy example of a problem one has to solve for magnetic resonance imaging (MRI). This technique makes use of the absorption and emission of energy in the radio frequency range of the electromagnetic spectrum.



Assume we have 4 hydrogen atoms, whose nuclei are excited with energy intensity a, b, c, d . We measure the spin echo in 4 different directions. $3 = a + b, 7 = c + d, 5 = a + c$ and $5 = b + d$. What is a, b, c, d ? Solution: $a = 2, b = 1, c = 3, d = 4$. However, also $a = 0, b = 3, c = 5, d = 2$ solves the problem. This system has not a unique solution even so there are 4 equations and 4 unknowns. A good introduction to MRI can be found online at (<http://www.cis.rit.edu/htbooks/mri/inside.htm>).

INCONSISTENT. $x - y = 4, y + z = 5, x + z = 6$ is a system with no solutions. It is called **inconsistent**.

EQUILIBRIUM. As an example of a system with many variables, consider a drum modeled by a fine net. The heights at each interior node needs the average of the heights of the 4 neighboring nodes. The height at the boundary is fixed. With n^2 nodes in the interior, we have to solve a system of n^2 equations. For example, for $n = 2$ (see left), the $n^2 = 4$ equations are $4x_{11} = a_{21} + a_{12} + x_{21} + x_{12}, 4x_{12} = x_{11} + x_{13} + x_{22} + x_{22}, 4x_{21} = x_{31} + x_{11} + x_{22} + a_{43}, 4x_{22} = x_{12} + x_{21} + a_{43} + a_{34}$. To the right, we see the solution to a problem with $n = 300$, where the computer had to solve a system with 90'000 variables.



LINEAR OR NONLINEAR?

- The ideal gas law** $PV = nKT$ for the P, V, T , the pressure p , volume V and temperature T of a gas.
- The Hook law** $F = k(x - a)$ relates the force F pulling a string extended to length x .
- Einstein's mass-energy equation** $E = mc^2$ relates restmass m with the energy E of a body.

ON THE HISTORY. In 2000 BC the Babylonians already studied problems which led to linear equations. In 200 BC, the Chinese used a method similar to Gaussian elimination to solve systems of linear equations.

MATRICES AND GAUSS-JORDAN

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MATRIX FORMULATION. Consider the system of linear equations like It can be written as $A\vec{x} = \vec{b}$, where A is a **matrix** called **coefficient matrix** and **column vectors** \vec{x} and \vec{b} .

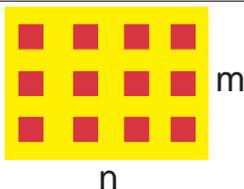
$$\begin{cases} 3x - y - z = 0 \\ -x + 2y - z = 0 \\ -x - y + 3z = 9 \end{cases} \quad A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \end{bmatrix}, \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 9 \end{bmatrix}.$$

$((A\vec{x})_i$ is the dot product of the i 'th row with \vec{x}).

We also look at the **augmented matrix** where the last column is separated for clarity reasons.

$$B = \left[\begin{array}{ccc|c} 3 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & 9 \end{array} \right].$$

MATRIX "JARGON". A rectangular array of numbers is called a **matrix**. If the matrix has m rows and n columns, it is called a $m \times n$ matrix. A matrix with one column only is called a **column vector**, a matrix with one row a **row vector**. The entries of a matrix are denoted by a_{ij} , where i is the row and j is the column. In the case of the linear equation above, the matrix A was a 3×3 square matrix and the augmented matrix B above is a 3×4 matrix.



GAUSS-JORDAN ELIMINATION. Gauss-Jordan Elimination is a process, where successive subtraction of multiples of other rows or scaling brings the matrix into **reduced row echelon form**. The elimination process consists of three possible steps which are called **elementary row operations**:

Swap two rows.

Scale a row

Subtract a multiple of a row from an other.

The process transfers a given matrix A into a new matrix $\text{rref}(A)$

REDUCED ECHELON FORM. A matrix is called in **reduced row echelon form**

- 1) if a row has nonzero entries, then the first nonzero entry is 1.
- 2) if a column contains a leading 1, then the other column entries are 0.
- 3) if a row has a leading 1, then every row above has a leading 1 to the left.

Pro memoriam: **Leaders like to be first, alone of their kind and other leaders above them to their left.**

RANK. The number of leading 1 in $\text{rref}(A)$ is called the rank of A .

SOLUTIONS OF LINEAR EQUATIONS. If $Ax = b$ is a linear system of equations with m equations and n unknowns, then A is a $m \times n$ matrix. We see that there are the following three possibilities:

- **Exactly one solution.** There is a leading 1 in each row but not in the last row.
- **Zero solutions.** There is a leading 1 in the last row.
- **Infinitely many solutions.** There are rows without leading 1 and no leading 1 in last row.

JIUZHANG SUANSHU. The technique of successively eliminating variables from systems of linear equations is called **Gauss elimination** or **Gauss Jordan elimination** and appeared already in the Chinese manuscript "Jiuzhang Suanshu" ('Nine Chapters on the Mathematical art'). The manuscript appeared around 200 BC in the Han dynasty and was probably used as a textbook. For more history of Chinese Mathematics, see

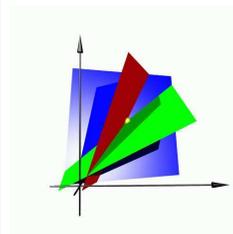
<http://aleph0.clarku.edu/~djoyce/mathhist/china.html>.



EXAMPLES. The reduced echelon form of the augmented matrix B determines on how many solutions the linear system $Ax = b$ has.



THE GOOD (1 solution)



$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & 2 \\ 1 & -1 & 1 & 5 \\ 2 & 1 & -1 & -2 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 2 & 1 & -1 & -2 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 3 & -3 & -12 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & -1 & -4 \end{array} \right]$$

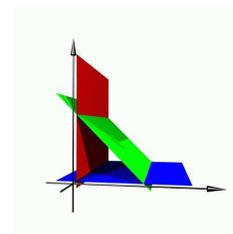
$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -3 & -6 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

$\text{rank}(A) = 3, \text{rank}(B) = 3.$

THE BAD (0 solution)



$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & 2 \\ 1 & -1 & 1 & 5 \\ 1 & 0 & 3 & -2 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 1 & 0 & 3 & -2 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & -7 \end{array} \right]$$

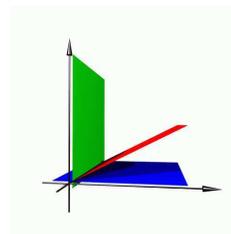
$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & -9 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & -9 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & -9 \end{array} \right]$$

$\text{rank}(A) = 2, \text{rank}(B) = 3.$

THE UGLY (∞ solutions)



$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & 2 \\ 1 & -1 & 1 & 5 \\ 1 & 0 & 3 & 7 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 1 & 0 & 3 & 7 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 2 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$\text{rank}(A) = 2, \text{rank}(B) = 2.$

JORDAN. The German geodesist Wilhelm **Jordan** (1842-1899) applied the Gauss-Jordan method to finding squared errors to work on surveying. (An other "Jordan", the French Mathematician Camille Jordan (1838-1922) worked on linear algebra topics also (Jordan form) and is often mistakenly credited with the Gauss-Jordan process.)

GAUSS. Gauss developed Gaussian elimination around 1800 and used it to solve least squares problems in celestial mechanics and later in geodesic computations. In 1809, Gauss published the book "Theory of Motion of the Heavenly Bodies" in which he used the method for solving astronomical problems. One of Gauss successes was the prediction of an asteroid orbit using linear algebra.



CERES. On 1. January of 1801, the Italian astronomer Giuseppe Piazzi (1746-1826) discovered **Ceres**, the first and until 2001 the largest known asteroid in the solar system. (A new found object called 2001 KX76 is estimated to have a 1200 km diameter, half the size of Pluto) Ceres is a rock of 914 km diameter. (The pictures Ceres in infrared light). Gauss was able to predict the orbit of Ceres from a few observations. By parameterizing the orbit with parameters and solving a linear system of equations (similar to one of the homework problems, where you will fit a cubic curve from 4 observations), he was able to derive the orbit parameters.

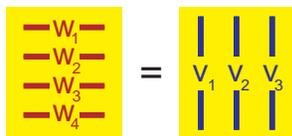


ON SOLUTIONS OF LINEAR EQUATIONS

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MATRIX. A rectangular array of numbers is called a **matrix**.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$



A matrix with n **rows** and m **columns** is called a $n \times m$ matrix. A matrix with one column is a **column vector**. The entries of a matrix are denoted a_{ij} , where i is the row number and j is the column number.

ROW AND COLUMN PICTURE. Two interpretations

$$A\vec{x} = \begin{bmatrix} -\vec{w}_1- \\ -\vec{w}_2- \\ \cdots \\ -\vec{w}_n- \end{bmatrix} \begin{bmatrix} | \\ | \\ \vec{x} \\ | \\ | \end{bmatrix} = \begin{bmatrix} \vec{w}_1 \cdot \vec{x} \\ \vec{w}_2 \cdot \vec{x} \\ \cdots \\ \vec{w}_n \cdot \vec{x} \end{bmatrix}$$



"Row and Column at Harvard"

$$A\vec{x} = \begin{bmatrix} | & | & \cdots & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_m\vec{v}_m = \vec{b}$$

Row picture: each b_i is the dot product of a row vector \vec{w}_i with \vec{x} .
Column picture: \vec{b} is a sum of scaled column vectors \vec{v}_j .

EXAMPLE. The system of linear equations

$$\begin{cases} 3x - 4y - 5z = 0 \\ -x + 2y - z = 0 \\ -x - y + 3z = 9 \end{cases}$$

is equivalent to $A\vec{x} = \vec{b}$, where A is a **coefficient matrix** and \vec{x} and \vec{b} are **vectors**.

$$A = \begin{bmatrix} 3 & -4 & -5 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \end{bmatrix}, \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 9 \end{bmatrix}$$

The **augmented matrix** (separators for clarity)

$$B = \left[\begin{array}{ccc|c} 3 & -4 & -5 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & 9 \end{array} \right]$$

In this case, the row vectors of A are

$$\vec{w}_1 = \begin{bmatrix} 3 & -4 & -5 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

The column vectors are

$$\vec{v}_1 = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -4 \\ 2 \\ -1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -5 \\ -1 \\ 3 \end{bmatrix}$$

Row picture:

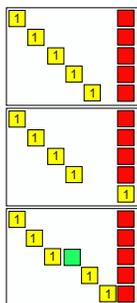
$$0 = b_1 = \begin{bmatrix} 3 & -4 & -5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Column picture:

$$\begin{bmatrix} 0 \\ 0 \\ 9 \end{bmatrix} = x_1 \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -4 \\ 2 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ -1 \\ 3 \end{bmatrix}$$

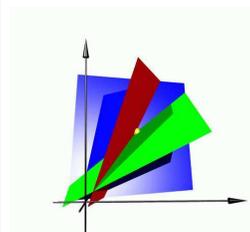
SOLUTIONS OF LINEAR EQUATIONS. A system $A\vec{x} = \vec{b}$ with n equations and m unknowns is defined by the $n \times m$ matrix A and the vector \vec{b} . The row reduced matrix $\text{rref}(B)$ of the augmented matrix $B = [A|\vec{b}]$ determines the number of solutions of the system $Ax = b$. The **rank** $\text{rank}(A)$ of a matrix A is the number of leading ones in $\text{rref}(A)$. There are three possibilities:

- **Consistent: Exactly one solution.** There is a leading 1 in each column of A but none in the last column of the augmented matrix B .
- **Inconsistent: No solutions.** There is a leading 1 in the last column of the augmented matrix B .
- **Consistent: Infinitely many solutions.** There are columns of A without leading 1.

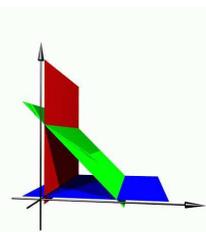


If $\text{rank}(A) = \text{rank}(A|\vec{b}) = m$, then there is **exactly 1 solution**.
 If $\text{rank}(A) < \text{rank}(A|\vec{b})$, there are **no solutions**.
 If $\text{rank}(A) = \text{rank}(A|\vec{b}) < m$: there are ∞ **solutions**.

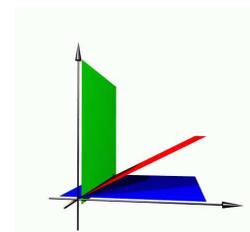
(exactly one solution)



(no solution)



(infinitely many solutions)

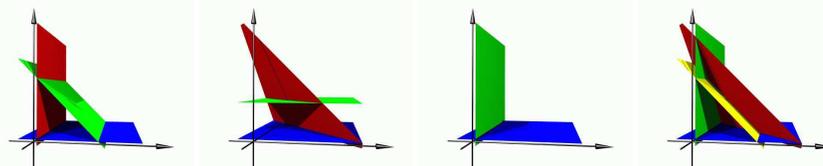


MURPHY'S LAW.

"If anything can go wrong, it will go wrong".
 "If you are feeling good, don't worry, you will get over it!"
 "For Gauss-Jordan elimination, the error happens early in the process and get unnoticed."



MURPHY'S LAW IS TRUE. Two equations could contradict each other. Geometrically, the two planes do not intersect. This is possible if they are parallel. Even without two planes being parallel, it is possible that there is no intersection between all three of them. It is also possible that not enough equations are at hand or that there are many solutions. Furthermore, there can be too many equations and the planes do not intersect.



RELEVANCE OF EXCEPTIONAL CASES. There are important applications, where "unusual" situations happen: For example in medical tomography, systems of equations appear which are "ill posed". In this case one has to be careful with the method.

The linear equations are then obtained from a method called the **Radon transform**. The task for finding a good method had led to a Nobel prize in Medicine 1979 for Allan Cormack. Cormack had sabbaticals at Harvard and probably has done part of his work on tomography here. Tomography helps today for example for cancer treatment.



MATRIX ALGEBRA I. Matrices can be added, subtracted if they have the same size:

$$A+B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix}$$

They can also be scaled by a scalar λ :

$$\lambda A = \lambda \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \cdots & \lambda a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda a_{m1} & \lambda a_{m2} & \cdots & \lambda a_{mn} \end{bmatrix}$$

LINEAR TRANSFORMATIONS

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TRANSFORMATIONS. A **transformation** T from a set X to a set Y is a rule, which assigns to every element in X an element $y = T(x)$ in Y . One calls X the **domain** and Y the **codomain**. A transformation is also called a **map** from X to Y .

LINEAR TRANSFORMATION. A map T from \mathbf{R}^n to \mathbf{R}^m is called a **linear transformation** if there is a $m \times n$ matrix A such that $T(\vec{x}) = A\vec{x}$.

EXAMPLES.

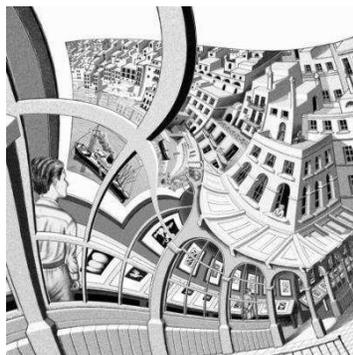
- To the linear transformation $T(x, y) = (3x + 4y, x + 5y)$ belongs the matrix $\begin{bmatrix} 3 & 4 \\ 1 & 5 \end{bmatrix}$. This transformation maps the plane onto itself.
- $T(x) = -13x$ is a linear transformation from the real line onto itself. The matrix is $A = [-13]$.
- To $T(\vec{x}) = \vec{y} \cdot \vec{x}$ from \mathbf{R}^3 to \mathbf{R} belongs the matrix $A = \vec{y} = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}$. This 1×3 matrix is also called a **row vector**. If the codomain is the real axes, one calls the map also a **function**. function defined on space.
- $T(x) = x\vec{y}$ from \mathbf{R} to \mathbf{R}^3 . $A = \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ is a 3×1 matrix which is also called a **column vector**. The map defines a line in space.
- $T(x, y, z) = (x, y)$ from \mathbf{R}^3 to \mathbf{R}^2 , A is the 2×3 matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. The map projects space onto a plane.
- To the map $T(x, y) = (x + y, x - y, 2x - 3y)$ belongs the 3×2 matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & -3 \end{bmatrix}$. The image of the map is a plane in three dimensional space.
- If $T(\vec{x}) = \vec{x}$, then T is called the **identity transformation**.

PROPERTIES OF LINEAR TRANSFORMATIONS. $T(\vec{0}) = \vec{0}$ $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ $T(\lambda\vec{x}) = \lambda T(\vec{x})$

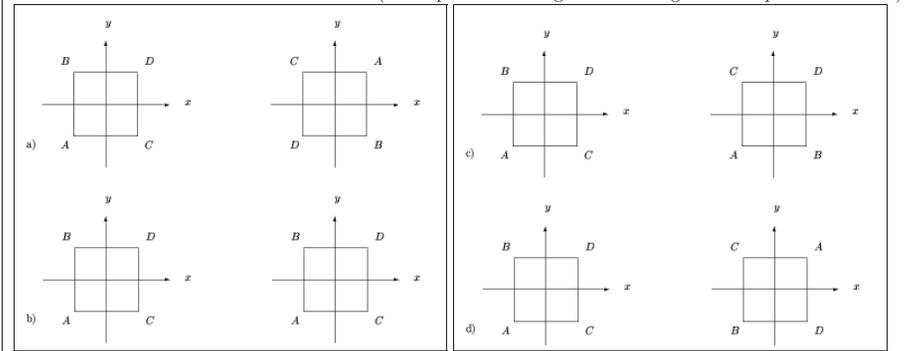
In words: Linear transformations are compatible with addition and scalar multiplication. It does not matter, whether we add two vectors before the transformation or add the transformed vectors.

ON LINEAR TRANSFORMATIONS. Linear transformations generalize the scaling transformation $x \mapsto ax$ in one dimensions. They are important in

- geometry (i.e. rotations, dilations, projections or reflections)
- art (i.e. perspective, coordinate transformations),
- CAD applications (i.e. projections),
- physics (i.e. Lorentz transformations),
- dynamics (linearizations of general maps are linear maps),
- compression (i.e. using Fourier transform or Wavelet transform),
- coding (many codes are linear codes),
- probability (i.e. Markov processes).
- linear equations (inversion is solving the equation)



LINEAR TRANSFORMATION OR NOT? (The square to the right is the image of the square to the left):



COLUMN VECTORS. A linear transformation $T(x) = Ax$ with $A = \begin{bmatrix} | & | & \cdots & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & \cdots & | \end{bmatrix}$ has the property that the column vector $\vec{v}_1, \vec{v}_2, \vec{v}_n$ are the images of the **standard vectors** $\vec{e}_1 = \begin{bmatrix} 1 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$, $\vec{e}_i = \begin{bmatrix} \cdot \\ \cdot \\ 1 \\ \cdot \\ 0 \end{bmatrix}$, $\vec{e}_n = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}$.

In order to find the matrix of a linear transformation, look at the image of the standard vectors and use those to build the columns of the matrix.

QUIZ.

- Find the matrix belonging to the linear transformation, which rotates a cube around the diagonal $(1, 1, 1)$ by 120 degrees $(2\pi/3)$.
- Find the linear transformation, which reflects a vector at the line containing the vector $(1, 1, 1)$.

INVERSE OF A TRANSFORMATION. If S is a second transformation such that $S(T\vec{x}) = \vec{x}$, for every \vec{x} , then S is called the **inverse** of T . We will discuss inverse transformations later in more detail.

SOLVING A LINEAR SYSTEM OF EQUATIONS. $A\vec{x} = \vec{b}$ means to invert the linear transformation $\vec{x} \mapsto A\vec{x}$. If the linear system has exactly one solution, then an inverse exists. We will write $\vec{x} = A^{-1}\vec{b}$ and see that the inverse of a linear transformation is again a linear transformation.

THE BRETSCHER CODE. Otto Bretschers book contains as a motivation a "code", where the encryption happens with the linear map $T(x, y) = (x + 3y, 2x + 5y)$. The map has the inverse $T^{-1}(x, y) = (-5x + 3y, 2x - y)$.



Cryptologists use often the following approach to crack a encryption. If one knows the input and output of some data, one often can decode the key. Assume we know, the enemy uses a Bretscher code and we know that $T(1, 1) = (3, 5)$ and $T(2, 1) = (7, 5)$. How do we get the code? The problem is to find the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

2x2 MATRIX. It is useful to decode the Bretscher code in general. If $ax + by = X$ and $cx + dy = Y$, then $x = (dX - bY)/(ad - bc)$, $y = (cX - aY)/(ad - bc)$. This is a linear transformation with matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

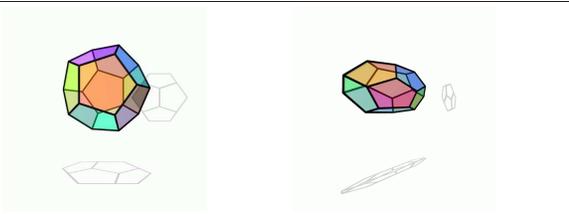
and the corresponding matrix is $A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} / (ad - bc)$.

"Switch diagonally, negate the wings and scale with a cross".

LINEAR TRAFOS IN GEOMETRY

Math 21b, O. Knill

LINEAR TRANSFORMATIONS DEFORMING A BODY



A CHARACTERIZATION OF LINEAR TRANSFORMATIONS: a transformation T from \mathbf{R}^n to \mathbf{R}^m which satisfies $T(\vec{0}) = \vec{0}$, $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ and $T(\lambda\vec{x}) = \lambda T(\vec{x})$ is a linear transformation.
Proof. Call $\vec{v}_i = T(\vec{e}_i)$ and define $S(\vec{x}) = A\vec{x}$. Then $S(\vec{e}_i) = T(\vec{e}_i)$. With $\vec{x} = x_1\vec{e}_1 + \dots + x_n\vec{e}_n$, we have $T(\vec{x}) = T(x_1\vec{e}_1 + \dots + x_n\vec{e}_n) = x_1\vec{v}_1 + \dots + x_n\vec{v}_n$ as well as $S(\vec{x}) = A(x_1\vec{e}_1 + \dots + x_n\vec{e}_n) = x_1\vec{v}_1 + \dots + x_n\vec{v}_n$ proving $T(\vec{x}) = S(\vec{x}) = A\vec{x}$.

SHEAR:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{[Diagram: A red square is transformed into a parallelogram tilted to the right.]} \quad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{[Diagram: A red square is transformed into a parallelogram tilted upwards.]}$$

In general, shears are transformation in the plane with the property that there is a vector \vec{w} such that $T(\vec{w}) = \vec{w}$ and $T(\vec{x}) - \vec{x}$ is a multiple of \vec{w} for all \vec{x} . If \vec{u} is orthogonal to \vec{w} , then $T(\vec{x}) = \vec{x} + (\vec{u} \cdot \vec{x})\vec{w}$.

SCALING:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{[Diagram: A red square is scaled up to a larger red square.]} \quad A = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \quad \text{[Diagram: A red square is scaled down to a smaller red square.]}$$

One can also look at transformations which scale x differently than y and where A is a diagonal matrix.

REFLECTION:

$$A = \begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix} \quad \text{[Diagram: A red square is reflected across a line at angle alpha.]} \quad A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{[Diagram: A red square is reflected across the x-axis.]}$$

Any reflection at a line has the form of the matrix to the left. A reflection at a line containing a unit vector \vec{u} is $T(\vec{x}) = 2(\vec{x} \cdot \vec{u})\vec{u} - \vec{x}$ with matrix $A = \begin{bmatrix} 2u_1^2 - 1 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2 - 1 \end{bmatrix}$

PROJECTION:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{[Diagram: A red square is projected onto the x-axis.]} \quad A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{[Diagram: A red square is projected onto the y-axis.]}$$

A projection onto a line containing unit vector \vec{u} is $T(\vec{x}) = (\vec{x} \cdot \vec{u})\vec{u}$ with matrix $A = \begin{bmatrix} u_1u_1 & u_2u_1 \\ u_1u_2 & u_2u_2 \end{bmatrix}$

ROTATION:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{[Diagram: A red square is rotated 180 degrees.]} \quad A = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \quad \text{[Diagram: A red square is rotated counter-clockwise by angle alpha.]}$$

Any rotation has the form of the matrix to the right.

ROTATION-DILATION:

$$A = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} \quad \text{[Diagram: A red square is rotated and dilated.]} \quad A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \quad \text{[Diagram: A red square is rotated and dilated by a factor of sqrt(a^2+b^2).]}$$

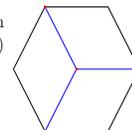
A rotation dilation is a composition of a rotation by angle $\arctan(y/x)$ and a dilation by a factor $\sqrt{x^2 + y^2}$. If $z = x + iy$ and $w = a + ib$ and $T(x, y) = (X, Y)$, then $X + iY = zw$. So a rotation dilation is tied to the process of the multiplication with a complex number.

BOOST:

$$A = \begin{bmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{bmatrix} \quad \text{[Diagram: A red square is transformed into a parallelogram, representing a Lorentz boost.]} \quad \text{The boost is a basic Lorentz transformation in special relativity. It acts on vectors } (x, ct), \text{ where } t \text{ is time, } c \text{ is the speed of light and } x \text{ is space.}$$

Unlike in Galileo transformation $(x, t) \mapsto (x + vt, t)$ (which is a shear, time t also changes during the transformation. The transformation has the effect that it changes length (Lorentz contraction). The angle α is related to v by $\tanh(\alpha) = v/c$. One can write also $A(x, ct) = ((x + vt)/\gamma, t + (v/c^2)\gamma x)$, with $\gamma = \sqrt{1 - v^2/c^2}$.

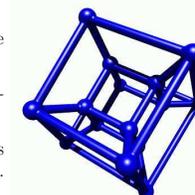
ROTATION IN SPACE. Rotations in space are defined by an axes of rotation and an angle. A rotation by 120° around a line containing $(0, 0, 0)$ and $(1, 1, 1)$ belongs to $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ which permutes $\vec{e}_1 \rightarrow \vec{e}_2 \rightarrow \vec{e}_3$.



REFLECTION AT PLANE. To a reflection at the xy -plane belongs the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ as can be seen by looking at the images of \vec{e}_i . The picture to the right shows the textbook and reflections of it at two different mirrors.



PROJECTION ONTO SPACE. To project a 4d-object into xyz-space, use for example the matrix $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The picture shows the projection of the four dimensional cube (tesseract, hypercube) with 16 edges $(\pm 1, \pm 1, \pm 1, \pm 1)$. The tesseract is the theme of the horror movie "hypercube".



THE INVERSE

Math 21b, O. Knill

INVERTIBLE TRANSFORMATIONS. A map T from X to Y is **invertible** if there is for every $y \in Y$ a **unique** point $x \in X$ such that $T(x) = y$.



EXAMPLES.

- 1) $T(x) = x^3$ is invertible from $X = \mathbf{R}$ to $X = Y$.
- 2) $T(x) = x^2$ is not invertible from $X = \mathbf{R}$ to $X = Y$.
- 3) $T(x, y) = (x^2 + 3x - y, x)$ is invertible from $X = \mathbf{R}^2$ to $Y = \mathbf{R}^2$.
- 4) $T(\vec{x}) = A\vec{x}$ linear and $\text{rref}(A)$ has an empty row, then T is not invertible.
- 5) If $T(\vec{x}) = A\vec{x}$ is linear and $\text{ref}(A) = \mathbf{1}_n$, then T is invertible.

INVERSE OF LINEAR TRANSFORMATION. If A is a $n \times n$ matrix and $T: \vec{x} \mapsto A\vec{x}$ has an inverse S , then S is linear. The matrix A^{-1} belonging to $S = T^{-1}$ is called the **inverse matrix** of A .

First proof: check that S is linear using the characterization $S(\vec{a} + \vec{b}) = S(\vec{a}) + S(\vec{b}), S(\lambda\vec{a}) = \lambda S(\vec{a})$ of linearity. Second proof: construct the inverse using Gauss-Jordan elimination.

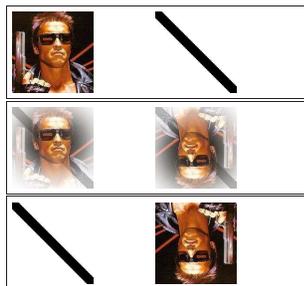
FINDING THE INVERSE. Let $\mathbf{1}_n$ be the $n \times n$ identity matrix. Start with $[A|\mathbf{1}_n]$ and perform Gauss-Jordan elimination. Then

$$\text{rref}([A|\mathbf{1}_n]) = [\mathbf{1}_n|A^{-1}]$$

Proof. The elimination process solves $A\vec{x} = \vec{e}_i$ simultaneously. This leads to solutions \vec{v}_i which are the columns of the inverse matrix A^{-1} because $A^{-1}\vec{e}_i = \vec{v}_i$.

EXAMPLE. Find the inverse of $A = \begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix}$.

$$\begin{array}{l} \left[\begin{array}{cc|cc} 2 & 6 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array} \right] \quad [A \mid \mathbf{1}_2] \\ \left[\begin{array}{cc|cc} 1 & 3 & 1/2 & 0 \\ 1 & 4 & 0 & 1 \end{array} \right] \quad [\dots \mid \dots] \\ \left[\begin{array}{cc|cc} 1 & 3 & 1/2 & 0 \\ 0 & 1 & -1/2 & 1 \end{array} \right] \quad [\dots \mid \dots] \\ \left[\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & 1 & -1/2 & 1 \end{array} \right] \quad [\mathbf{1}_2 \mid A^{-1}] \end{array}$$



The inverse is $A^{-1} = \begin{bmatrix} 2 & -3 \\ -1/2 & 1 \end{bmatrix}$.

THE INVERSE OF LINEAR MAPS $\mathbf{R}^2 \mapsto \mathbf{R}^2$:

If $ad - bc \neq 0$, the inverse of a linear transformation $\vec{x} \mapsto A\vec{x}$ with $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by the matrix $A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} / (ad - bc)$.

SHEAR:

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

DIAGONAL:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$$

REFLECTION:

$$A = \begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix}$$

$$A^{-1} = A = \begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix}$$

ROTATION:

$$A = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

ROTATION-DILATION:

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} a/r^2 & b/r^2 \\ -b/r^2 & a/r^2 \end{bmatrix}, r^2 = a^2 + b^2$$

BOOST:

$$A = \begin{bmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \cosh(\alpha) & -\sinh(\alpha) \\ -\sinh(\alpha) & \cosh(\alpha) \end{bmatrix}$$

NONINVERTIBLE EXAMPLE. The projection $\vec{x} \mapsto A\vec{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is a non-invertible transformation.

MORE ON SHEARS. The shears $T(x, y) = (x + ay, y)$ or $T(x, y) = (x, y + ax)$ in \mathbf{R}^2 can be generalized. A shear is a linear transformation which fixes some line L through the origin and which has the property that $T(\vec{x}) - \vec{x}$ is parallel to L for all \vec{x} . Shears are invertible.

PROBLEM. $T(x, y) = (3x/2 + y/2, y/2 - x/2)$ is a shear along a line L . Find L .

SOLUTION. Solve the system $T(x, y) = (x, y)$. You find that the vector $(1, -1)$ is preserved.

MORE ON PROJECTIONS. A linear map T with the property that $T(T(x)) = T(x)$ is a projection. Examples: $T(\vec{x}) = (\vec{y} \cdot \vec{x})\vec{y}$ is a projection onto a line spanned by a unit vector \vec{y} .

WHERE DO PROJECTIONS APPEAR? CAD: describe 3D objects using projections. A photo of an image is a projection. Compression algorithms like JPG or MPG or MP3 use projections where the high frequencies are cut away.

MORE ON ROTATIONS. A linear map T which preserves the angle between two vectors and the length of each vector is called a **rotation**. Rotations form an important class of transformations and will be treated later in more detail. In two dimensions, every rotation is of the form $x \mapsto A(x)$ with $A = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$.

An example of a rotations in three dimensions are $\vec{x} \mapsto A\vec{x}$, with $A = \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$. It is a rotation around the z axis.

MORE ON REFLECTIONS. Reflections are linear transformations different from the identity which are equal to their own inverse. Examples:

2D reflections at the origin: $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, **2D reflections at a line** $A = \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix}$.

3D reflections at origin: $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. **3D reflections at a line** $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. By

the way: in any dimensions, to a reflection at the line containing the unit vector \vec{u} belongs the matrix $[A]_{ij} = 2(u_i u_j) - [\mathbf{1}]_{ij}$, because $[B]_{ij} = u_i u_j$ is the matrix belonging to the projection onto the line.

The reflection at a line containing the unit vector $\vec{u} = [u_1, u_2, u_3]$ is $A = \begin{bmatrix} u_1^2 - 1 & u_1 u_2 & u_1 u_3 \\ u_2 u_1 & u_2^2 - 1 & u_2 u_3 \\ u_3 u_1 & u_3 u_2 & u_3^2 - 1 \end{bmatrix}$.

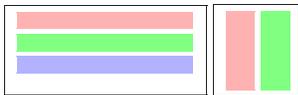
3D reflection at a plane $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

Reflections are important symmetries in physics: T (time reflection), P (reflection at a mirror), C (change of charge) are reflections. It seems today that the composition of TCP is a fundamental symmetry in nature.

MATRIX PRODUCT

Math 21b, O. Knill

MATRIX PRODUCT. If A is a $n \times m$ matrix and A is a $m \times p$ matrix, then AB is defined as the $n \times p$ matrix with entries $(BA)_{ij} = \sum_{k=1}^m B_{ik}A_{kj}$.



EXAMPLE. If B is a 3×4 matrix, and A is a 4×2 matrix then BA is a 3×2 matrix.

$$B = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 1 & 8 & 1 \\ 1 & 0 & 9 & 2 \end{bmatrix}, A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, BA = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 1 & 8 & 1 \\ 1 & 0 & 9 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 15 & 13 \\ 14 & 11 \\ 10 & 5 \end{bmatrix}.$$

COMPOSING LINEAR TRANSFORMATIONS. If $S: \mathbf{R}^m \rightarrow \mathbf{R}^p, x \mapsto Ax$ and $T: \mathbf{R}^p \rightarrow \mathbf{R}^n, y \mapsto By$ are linear transformations, then their composition $T \circ S: x \mapsto B(A(x))$ is a linear transformation from \mathbf{R}^m to \mathbf{R}^n . The corresponding $n \times m$ matrix is the matrix product BA .

EXAMPLE. Find the matrix which is a composition of a rotation around the x -axes by an angle $\pi/2$ followed by a rotation around the z -axes by an angle $\pi/2$.

SOLUTION. The first transformation has the property that $e_1 \rightarrow e_1, e_2 \rightarrow e_3, e_3 \rightarrow -e_2$, the second $e_1 \rightarrow e_2, e_2 \rightarrow -e_1, e_3 \rightarrow e_3$. If A is the matrix belonging to the first transformation and B the second, then BA is the matrix to the composition. The composition maps $e_1 \rightarrow -e_2 \rightarrow e_3 \rightarrow e_1$ is a rotation around a long diagonal. $B = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, BA = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

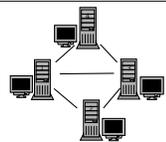
EXAMPLE. A rotation dilation is the composition of a rotation by $\alpha = \arctan(b/a)$ and a dilation (=scale) by $r = \sqrt{a^2 + b^2}$.

REMARK. Matrix multiplication can be seen a generalization of usual multiplication of numbers and also generalizes the dot product.

MATRIX ALGEBRA. Note that $AB \neq BA$ in general and A^{-1} does not always exist, otherwise, the same rules apply as for numbers: $A(BC) = (AB)C, AA^{-1} = A^{-1}A = 1_n, (AB)^{-1} = B^{-1}A^{-1}, A(B+C) = AB+AC, (B+C)A = BA+CA$ etc.

PARTITIONED MATRICES. The entries of matrices can themselves be matrices. If B is a $n \times p$ matrix and A is a $p \times m$ matrix, and assume the entries are $k \times k$ matrices, then BA is a $n \times m$ matrix, where each entry $(BA)_{ij} = \sum_{l=1}^p B_{il}A_{lj}$ is a $k \times k$ matrix. Partitioning matrices can be useful to improve the speed of matrix multiplication

EXAMPLE. If $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \\ A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}$, where A_{ij} are $k \times k$ matrices with the property that A_{11} and A_{22} are invertible, then $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ is the inverse of A .

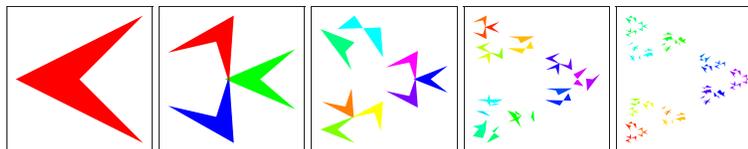


NETWORKS. Let us associate to the computer network a matrix

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \text{ A worm in the first computer is associated to } \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \text{ The}$$

vector Ax has a 1 at the places, where the worm could be in the next step. The vector $(AA)(x)$ tells, in how many ways the worm can go from the first computer to other hosts in 2 steps. In our case, it can go in three different ways back to the computer itself.

Matrices help to solve combinatorial problems (see movie "Good will hunting"). For example, what does $[A^{1000}]_{22}$ tell about the worm infection of the network? What does it mean if A^{100} has no zero entries?



FRACTALS. Closely related to linear maps are **affine maps** $x \mapsto Ax + b$. They are compositions of a linear map with a translation. It is **not** a linear map if $B(0) \neq 0$. Affine maps can be disguised as linear maps in the following way: let $y = \begin{bmatrix} x \\ 1 \end{bmatrix}$ and define the $(n+1) \times (n+1)$ matrix $B = \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix}$. Then $By = \begin{bmatrix} Ax + b \\ 1 \end{bmatrix}$.

Fractals can be constructed by taking for example 3 affine maps R, S, T which contract space. For a given object Y_0 define $Y_1 = R(Y_0) \cup S(Y_0) \cup T(Y_0)$ and recursively $Y_k = R(Y_{k-1}) \cup S(Y_{k-1}) \cup T(Y_{k-1})$. The above picture shows Y_k after some iterations. In the limit, for example if $R(Y_0), S(Y_0)$ and $T(Y_0)$ are disjoint, the sets Y_k converge to a **fractal**, an object with dimension strictly between 1 and 2.

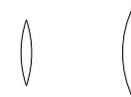


CHAOS. Consider a map in the plane like $T: \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2x + 2\sin(x) - y \\ x \end{bmatrix}$. We apply this map again and again and follow the points $(x_1, y_1) = T(x, y), (x_2, y_2) = T(T(x, y))$, etc. Lets write T^n for the n -th iteration of the map and (x_n, y_n) for the image of (x, y) under the map T^n . The linear approximation of the map at a point (x, y) is the matrix $DT(x, y) = \begin{bmatrix} 2 + 2\cos(x) - 1 & -1 \\ 1 & 0 \end{bmatrix}$. (If $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix}$, then the row vectors of $DT(x, y)$ are just the gradients of f and g). T is called **chaotic** at (x, y) , if the entries of $D(T^n)(x, y)$ grow exponentially fast with n . By the **chain rule**, $D(T^n)$ is the product of matrices $DT(x_i, y_i)$. For example, T is chaotic at $(0, 0)$. If there is a positive probability to hit a chaotic point, then T is called chaotic.

FALSE COLORS. Any color can be represented as a vector (r, g, b) , where $r \in [0, 1]$ is the red $g \in [0, 1]$ is the green and $b \in [0, 1]$ is the blue component. Changing colors in a picture means applying a transformation on the cube. Let $T: (r, g, b) \mapsto (g, b, r)$ and $S: (r, g, b) \mapsto (r, g, 0)$. What is the composition of these two linear maps?



OPTICS. Matrices help to calculate the motion of light rays through lenses. A light ray $y(s) = x + ms$ in the plane is described by a vector (x, m) . Following the light ray over a distance of length L corresponds to the map $(x, m) \mapsto (x + mL, m)$. In the lens, the ray is bent depending on the height x . The transformation in the lens is $(x, m) \mapsto (x, m - kx)$, where k is the strength of the lens.



$$\begin{bmatrix} x \\ m \end{bmatrix} \mapsto A_L \begin{bmatrix} x \\ m \end{bmatrix} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ m \end{bmatrix}, \begin{bmatrix} x \\ m \end{bmatrix} \mapsto B_k \begin{bmatrix} x \\ m \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix} \begin{bmatrix} x \\ m \end{bmatrix}.$$

Examples:

- 1) Eye looking far: $A_R B_k$.
- 2) Eye looking at distance L : $A_R B_k A_L$.
- 3) Telescope: $B_{k_2} A_L B_{k_1}$. (More about it in problem 80 in section 2.4).

Error correcting code	Math21b, O.Knill
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Let's try out the error correcting code as in the book (problem 53-54 in 3.1) which appears also in the homework.

I) Encoding.

To do so, we encode the letters of the alphabet by pairs of three vectors containing zeros and ones:

- | | | |
|----------------------------------|----------------------------------|----------------------------------|
| $A = (0, 0, 0, 1), (0, 0, 0, 1)$ | $B = (0, 0, 0, 1), (0, 0, 1, 0)$ | $C = (0, 0, 0, 1), (0, 0, 1, 1)$ |
| $D = (0, 0, 0, 1), (0, 1, 0, 1)$ | $E = (0, 0, 0, 1), (0, 1, 1, 0)$ | $F = (0, 0, 0, 1), (0, 1, 1, 1)$ |
| $G = (0, 0, 0, 1), (1, 0, 0, 1)$ | $H = (0, 0, 0, 1), (1, 0, 1, 0)$ | $I = (0, 0, 0, 1), (1, 0, 1, 1)$ |
| $J = (0, 0, 0, 1), (1, 1, 0, 1)$ | $K = (0, 0, 0, 1), (1, 1, 1, 0)$ | $L = (0, 0, 0, 1), (1, 1, 1, 1)$ |
| $M = (0, 0, 1, 0), (0, 0, 0, 1)$ | $N = (0, 0, 1, 0), (0, 0, 1, 0)$ | $O = (0, 0, 1, 0), (0, 0, 1, 1)$ |
| $P = (0, 0, 1, 0), (0, 1, 0, 1)$ | $Q = (0, 0, 1, 0), (0, 1, 1, 0)$ | $R = (0, 0, 1, 0), (0, 1, 1, 1)$ |
| $S = (0, 0, 1, 0), (1, 0, 0, 1)$ | $T = (0, 0, 1, 0), (1, 0, 1, 0)$ | $U = (0, 0, 1, 0), (1, 0, 1, 1)$ |
| $V = (0, 0, 1, 0), (1, 1, 0, 1)$ | $W = (0, 0, 1, 0), (1, 1, 1, 0)$ | $X = (0, 0, 1, 0), (1, 1, 1, 1)$ |
| $Y = (0, 0, 1, 1), (1, 0, 0, 1)$ | $Z = (0, 0, 1, 1), (1, 0, 1, 0)$ | $? = (0, 0, 1, 1), (1, 0, 1, 1)$ |
| $! = (0, 0, 1, 1), (1, 0, 0, 1)$ | $. = (0, 0, 1, 1), (1, 0, 1, 0)$ | $, = (0, 0, 1, 1), (1, 0, 1, 1)$ |

Choose a letter .

Look up in the above table the pair (x, y) which belongs to this letter.

$$x = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}, \quad y = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

Now we build (Mx, My) , where M is the matrix $M = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Use $1 + 1 = 0$ in the matrix multiplications which follow! Let's go:

$$Mx = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

$$My = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

Now, fold this page at the prescribed fold and show the two vectors Mx, My to your friend, who writes down on the top of the second page.

II) Transmission.

You obtain now the vectors Mx, My from your neighbor. Copy the two vectors (Mx, My) of him or her but add one error. To do so, switch one 1 to 0 or one 0 to 1 in the above vectors.

$$u = Mx + e = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \quad v = My + f = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

III) Detect the error e and f.

Detect errors by forming

$$Hu = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

$$Hv = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

Now look in which column Hu or Hv is. Put 0's everywhere in e except at that place, where you put a 1. For example if Hu is the second column, then put a 1 at the second place. We obtain e and f :

$$e = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}, \quad f = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

IV) Decode the message.

Let $P = \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \end{bmatrix} = \begin{bmatrix} d \\ e \\ f \\ g \end{bmatrix}$. Determine $Pe = P \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$, $Pf = P \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$. In an error-free transmission (Pu, Pv) would give the right result back. Now

$$Pu = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}, \quad Pv = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

satisfy $Pu = x + Pe, Pv = y + Pf$. We recover the original message by subtracting Pe, Pf from that

$$x = Pu - Pe = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}, \quad y = Pv - Pf = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

(In the subtraction like addition, use $1 + 1 = 0, -1 - 1 = 0, 1 - 1 = 0, -1 = 1$.)

The letter belonging to (x, y) (look it up) is .

IMAGE AND KERNEL

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IMAGE. If $T : \mathbf{R}^m \rightarrow \mathbf{R}^n$ is a linear transformation, then $\{T(\vec{x}) \mid \vec{x} \in \mathbf{R}^m\}$ is called the **image** of T . If $T(\vec{x}) = A\vec{x}$, then the image of T is also called the image of A . We write $\text{im}(A)$ or $\text{im}(T)$.

EXAMPLES.

- 1) If $T(x, y, z) = (x, y, 0)$, then $T(\vec{x}) = A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. The image of T is the $x - y$ plane.
- 2) If $T(x, y) = (\cos(\phi)x - \sin(\phi)y, \sin(\phi)x + \cos(\phi)y)$ is a rotation in the plane, then the image of T is the whole plane.
- 3) If $T(x, y, z) = x + y + z$, then the image of T is \mathbf{R} .

SPAN. The **span** of vectors $\vec{v}_1, \dots, \vec{v}_k$ in \mathbf{R}^n is the set of all combinations $c_1\vec{v}_1 + \dots + c_k\vec{v}_k$, where c_i are real numbers.

PROPERTIES.

The image of a linear transformation $\vec{x} \mapsto A\vec{x}$ is the span of the column vectors of A .
The image of a linear transformation contains 0 and is closed under addition and scalar multiplication.

KERNEL. If $T : \mathbf{R}^m \rightarrow \mathbf{R}^n$ is a linear transformation, then the set $\{x \mid T(x) = 0\}$ is called the **kernel** of T . If $T(\vec{x}) = A\vec{x}$, then the kernel of T is also called the kernel of A . We write $\text{ker}(A)$ or $\text{ker}(T)$.

EXAMPLES. (The same examples as above)

- 1) The kernel is the z -axes. Every vector $(0, 0, z)$ is mapped to 0.
- 2) The kernel consists only of the point $(0, 0, 0)$.
- 3) The kernel consists of all vector (x, y, z) for which $x + y + z = 0$. The kernel is a plane.

PROPERTIES.

The kernel of a linear transformation contains 0 and is closed under addition and scalar multiplication.

IMAGE AND KERNEL OF INVERTIBLE MAPS. A linear map $\vec{x} \mapsto A\vec{x}, \mathbf{R}^n \mapsto \mathbf{R}^n$ is invertible if and only if $\text{ker}(A) = \{0\}$ if and only if $\text{im}(A) = \mathbf{R}^n$.

HOW DO WE COMPUTE THE IMAGE? The column vectors of A span the image. We will see later that the columns with leading ones alone span already the image.

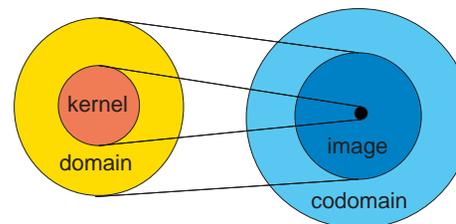
EXAMPLES. (The same examples as above)

- 1) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ span the image.
- 2) $\begin{bmatrix} \cos(\phi) \\ -\sin(\phi) \end{bmatrix}$ and $\begin{bmatrix} \sin(\phi) \\ \cos(\phi) \end{bmatrix}$ span the image.
- 3) The 1D vector $[1]$ spans the image.

HOW DO WE COMPUTE THE KERNEL? Just solve the linear system of equations $A\vec{x} = \vec{0}$. Form $\text{rref}(A)$. For every column without leading 1 we can introduce a **free variable** s_j . If \vec{x} is the solution to $A\vec{x}_i = 0$, where all s_j are zero except $s_i = 1$, then $\vec{x} = \sum_j s_j \vec{x}_j$ is a general vector in the kernel.

EXAMPLE. Find the kernel of the linear map $\mathbf{R}^3 \rightarrow \mathbf{R}^4, \vec{x} \mapsto A\vec{x}$ with $A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 5 \\ 3 & 9 & 1 \\ -2 & -6 & 0 \end{bmatrix}$. Gauss-Jordan

elimination gives: $B = \text{rref}(A) = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. We see one column without leading 1 (the second one). The equation $B\vec{x} = 0$ is equivalent to the system $x + 3y = 0, z = 0$. After fixing $z = 0$, can chose $y = t$ freely and obtain from the first equation $x = -3t$. Therefore, the kernel consists of vectors $t \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$. In the book, you have a detailed calculation, in a case, where the kernel is 2 dimensional.



WHY DO WE LOOK AT THE KERNEL?

- It is useful to understand linear maps. To which degree are they non-invertible?
- Helpful to understand quantitatively how many solutions a linear equation $Ax = b$ has. If x is a solution and y is in the kernel of A , then also $A(x + y) = b$, so that $x + y$ solves the system also.

WHY DO WE LOOK AT THE IMAGE?

- A solution $Ax = b$ can be solved if and only if b is in the image of A .
- Knowing about the kernel and the image is useful in the similar way that it is useful to know about the domain and range of a general map and to understand the graph of the map.

In general, the abstraction helps to understand topics like error correcting codes (Problem 53/54 in Bretschers book), where two matrices H, M with the property that $\text{ker}(H) = \text{im}(M)$ appear. The encoding $x \mapsto Mx$ is robust in the sense that adding an error e to the result $Mx \mapsto Mx + e$ can be corrected: $H(Mx + e) = He$ allows to find e and so Mx . This allows to recover $x = PMx$ with a projection P .

PROBLEM. Find $\text{ker}(A)$ and $\text{im}(A)$ for the 1×3 matrix $A = [5, 1, 4]$, a row vector.

ANSWER. $A \cdot \vec{x} = A\vec{x} = 5x + y + 4z = 0$ shows that the kernel is a plane with normal vector $[5, 1, 4]$ through the origin. The image is the codomain, which is \mathbf{R} .

PROBLEM. Find $\text{ker}(A)$ and $\text{im}(A)$ of the linear map $x \mapsto v \times x$, (the cross product with v).

ANSWER. The kernel consists of the line spanned by v , the image is the plane orthogonal to v .

PROBLEM. Fix a vector w in space. Find $\text{ker}(A)$ and image $\text{im}(A)$ of the linear map from \mathbf{R}^6 to \mathbf{R}^3 given by $x, y \mapsto [x, v, y] = (x \times y) \cdot w$.

ANSWER. The kernel consist of all (x, y) such that their cross product orthogonal to w . This means that the plane spanned by x, y contains w .

PROBLEM Find $\text{ker}(T)$ and $\text{im}(T)$ if T is a composition of a rotation R by 90 degrees around the z -axes with with a projection onto the x - z plane.

ANSWER. The kernel of the projection is the y axes. The x axes is rotated into the y axes and therefore the kernel of T . The image is the x - z plane.

PROBLEM. Can the kernel of a square matrix A be trivial if $A^2 = 0$, where 0 is the matrix containing only 0?

ANSWER. No: if the kernel were trivial, then A were invertible and A^2 were invertible and be different from 0 .

PROBLEM. Is it possible that a 3×3 matrix A satisfies $\text{ker}(A) = \mathbf{R}^3$ without $A = 0$?

ANSWER. No, if $A \neq 0$, then A contains a nonzero entry and therefore, a column vector which is nonzero.

PROBLEM. What is the kernel and image of a projection onto the plane $\Sigma : x - y + 2z = 0$?

ANSWER. The kernel consists of all vectors orthogonal to Σ , the image is the plane Σ .

PROBLEM. Given two square matrices A, B and assume $AB = BA$. You know $\text{ker}(A)$ and $\text{ker}(B)$. What can you say about $\text{ker}(AB)$?

ANSWER. $\text{ker}(A)$ is contained in $\text{ker}(BA)$. Similarly $\text{ker}(B)$ is contained in $\text{ker}(AB)$. Because $AB = BA$, the kernel of AB contains both $\text{ker}(A)$ and $\text{ker}(B)$. (It can be bigger as the example $A = B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ shows.)

PROBLEM. What is the kernel of the partitioned matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ if $\text{ker}(A)$ and $\text{ker}(B)$ are known?

ANSWER. The kernel consists of all vectors (\vec{x}, \vec{y}) , where $\vec{x} \in \text{ker}(A)$ and $\vec{y} \in \text{ker}(B)$.

BASIS

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LINEAR SUBSPACE: A subset X of \mathbf{R}^n which is closed under addition and scalar multiplication is called a **linear subspace** of \mathbf{R}^n . We have to check three conditions: (a) $0 \in V$, (b) $\vec{v} + \vec{w} \in V$ if $\vec{v}, \vec{w} \in V$. (c) $\lambda \vec{v} \in V$ if \vec{v} and λ is a real number.

WHICH OF THE FOLLOWING SETS ARE LINEAR SPACES?

- a) The kernel of a linear map.
- b) The image of a linear map.
- c) The upper half plane.
- d) The set $x^2 = y^2$.
- e) the line $x + y = 0$.
- f) The plane $x + y + z = 1$.
- g) The unit circle.
- h) The x axes.

BASIS. A set of vectors $\vec{v}_1, \dots, \vec{v}_m$ is a **basis** of a linear subspace X of \mathbf{R}^n if they are **linear independent** and if they **span** the space X . Linear independent means that there are no nontrivial **linear relations** $a_i \vec{v}_i + \dots + a_m \vec{v}_m = 0$. Spanning the space means that very vector \vec{v} can be written as a linear combination $\vec{v} = a_1 \vec{v}_1 + \dots + a_m \vec{v}_m$ of basis vectors.



EXAMPLE 1) The vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ form a basis in the three dimensional space.

If $\vec{v} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$, then $\vec{v} = \vec{v}_1 + 2\vec{v}_2 + 3\vec{v}_3$ and this representation is unique. We can find the coefficients by solving

$A\vec{x} = \vec{v}$, where A has the \vec{v}_i as column vectors. In our case, $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$ had the unique solution $x = 1, y = 2, z = 3$ leading to $\vec{v} = \vec{v}_1 + 2\vec{v}_2 + 3\vec{v}_3$.

- EXAMPLE 2) Two nonzero vectors in the plane which are not parallel form a basis.
- EXAMPLE 3) Four vectors in \mathbf{R}^3 are not a basis.
- EXAMPLE 4) Two vectors in \mathbf{R}^3 never form a basis.
- EXAMPLE 5) Three nonzero vectors in \mathbf{R}^3 which are not contained in a single plane form a basis in \mathbf{R}^3 .
- EXAMPLE 6) The columns of an invertible $n \times n$ matrix form a basis in \mathbf{R}^n .

FACT. If $\vec{v}_1, \dots, \vec{v}_n$ is a basis, then every vector \vec{v} can be represented **uniquely** as a linear combination of the basis vectors: $\vec{v} = a_1 \vec{v}_1 + \dots + a_n \vec{v}_n$.

REASON. There is at least one representation because the vectors \vec{v}_i span the space. If there were two different representations $\vec{v} = a_1 \vec{v}_1 + \dots + a_n \vec{v}_n$ and $\vec{v} = b_1 \vec{v}_1 + \dots + b_n \vec{v}_n$, then subtraction would lead to $0 = (a_1 - b_1)\vec{v}_1 + \dots + (a_n - b_n)\vec{v}_n$. Linear independence shows $a_1 - b_1 = a_2 - b_2 = \dots = a_n - b_n = 0$.

FACT. If n vectors $\vec{v}_1, \dots, \vec{v}_n$ span a space and $\vec{w}_1, \dots, \vec{w}_m$ are linear independent, then $m \leq n$.

REASON. This is intuitively clear in dimensions up to 3. You can not have 4 vectors in three dimensional space which are linearly independent. We will give a precise reason later.

A BASIS DEFINES AN INVERTIBLE MATRIX. The $n \times n$ matrix $A = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \\ | & | & & | \end{bmatrix}$ is invertible if and only if $\vec{v}_1, \dots, \vec{v}_n$ define a basis in \mathbf{R}^n .

EXAMPLE. In example 1), the 3×3 matrix A is invertible.

FINDING A BASIS FOR THE KERNEL. To solve $Ax = 0$, we bring the matrix A into the reduced row echelon form $\text{rref}(A)$. For every non-leading entry in $\text{rref}(A)$, we will get a **free variable** t_i . Writing the system $Ax = 0$ with these free variables gives us an equation $\vec{x} = \sum_i t_i \vec{v}_i$. The vectors \vec{v}_i form a basis of the kernel of A .

REMARK. The problem to find a basis for all vectors \vec{w}_i which are orthogonal to a given set of vectors, is equivalent to the problem to find a basis for the kernel of the matrix which has the vectors \vec{w}_i in its rows.

FINDING A BASIS FOR THE IMAGE. Bring the $m \times n$ matrix A into the form $\text{rref}(A)$. Call a column a **pivot column**, if it contains a leading 1. The corresponding set of column vectors of the original matrix A form a basis for the image because they are linearly independent and they all are in the image. The pivot columns also span the image because if we remove the nonpivot columns, and \vec{b} is in the image, we can solve $A\vec{x} = \vec{b}$.

EXAMPLE. $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 3 & 4 & 5 \end{bmatrix}$. has two pivot columns, the first and second one. For $\vec{b} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, we can solve $A\vec{x} = \vec{b}$. We can also solve $B\vec{x} = \vec{b}$ with $B = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$.

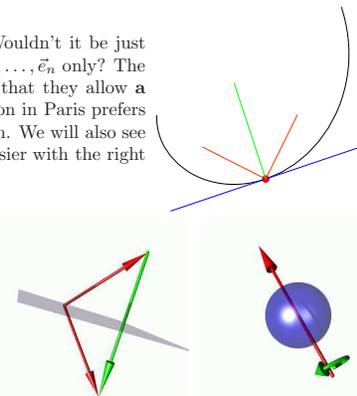
REMARK. The problem to find a basis of the subspace generated by $\vec{v}_1, \dots, \vec{v}_n$, is the problem to find a basis for the image of the matrix A with column vectors $\vec{v}_1, \dots, \vec{v}_n$.

EXAMPLE. Let A be the matrix $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. In reduced row echelon form is $B = \text{rref}(A) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

To determine a basis of the kernel we write $Bx = 0$ as a system of linear equations: $x + y = 0, z = 0$. The variable y is the free variable. With $y = t, x = -t$ is fixed. The linear system $\text{rref}(A)x = 0$ is solved by $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. So, $\vec{v} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ is a basis of the kernel.

EXAMPLE. Because the first and third vectors in $\text{rref}(A)$ are columns with leading 1's, the first and third columns $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ of A form a basis of the image of A .

WHY DO WE INTRODUCE BASIS VECTORS? Wouldn't it be just easier to always look at the standard basis vectors $\vec{e}_1, \dots, \vec{e}_n$ only? The reason for the need of more general basis vectors is that they allow a **more flexible adaptation** to the situation. A person in Paris prefers a different set of basis vectors than a person in Boston. We will also see that in many applications, problems can be solved easier with the right basis.



For example, to describe the reflection of a ray at a plane or at a curve, it is preferable to use basis vectors which are tangent or orthogonal to the plane. When looking at a rotation, it is good to have one basis vector in the axis of rotation, the other two orthogonal to the axis. Choosing the right basis will be especially important when studying differential equations.

A PROBLEM. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$. Find a basis for $\ker(A)$ and $\text{im}(A)$.

SOLUTION. From $\text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ we see that $\vec{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is in the kernel. The two column vectors

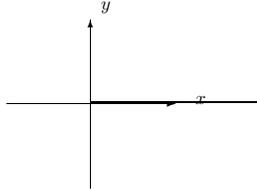
$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ of A form a basis of the image because the first and third column are pivot columns.

LINEAR SUBSPACES

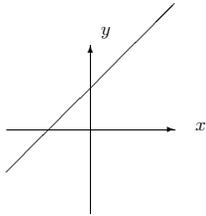
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Which of the following subsets in the plane are linear spaces?

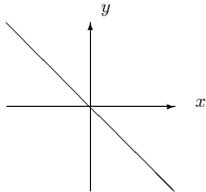
- Positive real axes.



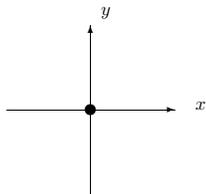
- The line $-x + y = 1$



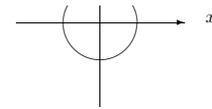
- The line $x - y = 0$



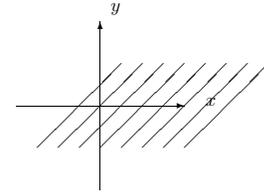
- The point $\{0\}$



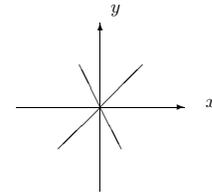
- The unit circle.



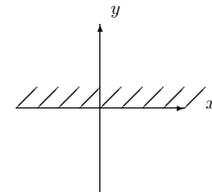
- The whole plane.



- The union of the two lines



- The upper half plane.



DIMENSION

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REVIEW LINEAR SUBSPACE $X \subset \mathbb{R}^n$ is a **linear space** if $\vec{0} \in X$ and if X is closed under addition and scalar multiplication. Examples are \mathbb{R}^n , $X = \ker(A)$, $X = \text{im}(A)$. In order to describe linear spaces, we need the notion of a basis:

REVIEW BASIS. $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\} \subset X$
 \mathcal{B} linear independent: $c_1\vec{v}_1 + \dots + c_n\vec{v}_n = 0$ implies $c_1 = \dots = c_n = 0$.
 \mathcal{B} span X : $\vec{v} \in X$ then $\vec{v} = a_1\vec{v}_1 + \dots + a_n\vec{v}_n$.
 \mathcal{B} basis: both linear independent and span.



BASIS: ENOUGH BUT NOT TOO MUCH. The spanning condition for a basis assures that there are **enough** vectors to represent any other vector, the linear independence condition assures that there are **not too many** vectors. A basis is, where J.Lo meets A.Hi: Left: J.Lopez in "Enough", right "The man who new **too much**" by A.Hitchcock



DIMENSION. The number of elements in a basis of X is independent of the choice of the basis. It is called the **dimension** of X .

UNIQUE REPRESENTATION. $\vec{v}_1, \dots, \vec{v}_n \in X$ **basis** \Rightarrow every $\vec{v} \in X$ can be written uniquely as a sum $\vec{v} = a_1\vec{v}_1 + \dots + a_n\vec{v}_n$.

EXAMPLES. The dimension of $\{0\}$ is zero. The dimension of any line is 1. The dimension of a plane is 2, the dimension of three dimensional space is 3. The dimension is independent on where the space is embedded in. For example: a line in the plane and a line in space have dimension 1.

REVIEW: KERNEL AND IMAGE. We can construct a basis of the kernel and image of a linear transformation $T(x) = Ax$ by forming $B = \text{rref}A$. The set of Pivot columns in A form a basis of the image of T , a basis for the kernel is obtained by solving $Bx = 0$ and introducing free variables for each non-pivot column.

PROBLEM. Find a basis of the span of the column vectors of A

$$A = \begin{bmatrix} 1 & 11 & 111 & 11 & 1 \\ 11 & 111 & 1111 & 111 & 11 \\ 111 & 1111 & 11111 & 1111 & 111 \end{bmatrix}$$

Find also a basis of the **row space** the span of the row vectors.

SOLUTION. In order to find a basis of the column space, we row reduce the matrix A and identify the leading 1: we have

$$\text{rref}(A) = \begin{bmatrix} \boxed{1} & 0 & -10 & 0 & 1 \\ 0 & \boxed{1} & 11 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Because the first two columns have leading $\boxed{1}$, the first two columns of A span the image of A , the column

space. The basis is $\left\{ \begin{bmatrix} 1 \\ 11 \\ 111 \end{bmatrix}, \begin{bmatrix} 11 \\ 1111 \end{bmatrix} \right\}$.

Now produce a matrix B which contains the rows of A as columns

$$B = \begin{bmatrix} 1 & 11 & 111 \\ 11 & 111 & 1111 \\ 111 & 1111 & 11111 \\ 1 & 11 & 111 \end{bmatrix}$$

and row reduce it to

$$\text{rref}(B) = \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The first two columns of A span the image of B . $\mathcal{B} =$

$$\left\{ \begin{bmatrix} 1 \\ 11 \\ 111 \\ 1 \end{bmatrix}, \begin{bmatrix} 11 \\ 1111 \\ 11111 \\ 11 \end{bmatrix} \right\}$$

LEMMA. If q vectors $\vec{w}_1, \dots, \vec{w}_q$ span X and $\vec{v}_1, \dots, \vec{v}_p$ are linearly independent in X , then $p \leq q$.

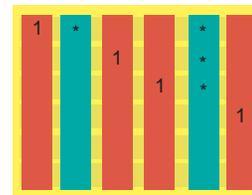
REASON (an other proof is in the book). Assume $q < p$. Because \vec{w}_i span, each vector \vec{v}_i can be written as $\sum_{j=1}^q a_{ij}\vec{w}_j = \vec{v}_i$. After doing Gauss-Jordan elimination of the augmented $(q \times (p+n))$ -matrix to

this system: $\left[\begin{array}{ccc|c} a_{11} & \dots & a_{1q} & \vec{w}_1^T \\ \dots & \dots & \dots & \dots \\ a_{p1} & \dots & a_{pq} & \vec{w}_q^T \end{array} \right]$, where \vec{w}^T is the vector \vec{v} written as a row vector. Note that each row of A of this $[A|b]$ contains some nonzero entry. We end up with a matrix, which contains a last line $| 0 \dots 0 | b_1\vec{w}_1^T + \dots + b_q\vec{w}_q^T$ showing that $b_1\vec{w}_1^T + \dots + b_q\vec{w}_q^T = 0$. Not all b_j are zero because we had to eliminate some nonzero entries in the last row of A . This nontrivial relation of \vec{w}_i^T (and the same relation for column vectors \vec{w}) is a contradiction to the linear independence of the \vec{w}_j . The assumption $q < p$ can not be true.

THEOREM. Given a basis $\mathcal{A} = \{v_1, \dots, v_n\}$ and a basis $\mathcal{B} = \{w_1, \dots, w_m\}$ of X , then $m = n$.

PROOF. Because \mathcal{A} spans X and \mathcal{B} is linearly independent, we know that $n \leq m$. Because \mathcal{B} spans X and \mathcal{A} is linearly independent also $m \leq n$ holds. Together, $n \leq m$ and $m \leq n$ implies $n = m$.

DIMENSION OF THE KERNEL. The number of columns in $\text{rref}(A)$ without leading 1's is the **dimension of the kernel** $\dim(\ker(A))$: we can introduce a parameter for each such column when solving $Ax = 0$ using Gauss-Jordan elimination. The dimension of the kernel of A is the number of "free variables" of A .



DIMENSION OF THE IMAGE. The number of **leading 1** in $\text{rref}(A)$, the rank of A is the **dimension of the image** $\dim(\text{im}(A))$ because every such leading 1 produces a different column vector (called **pivot column vectors**) and these column vectors are linearly independent.

DIMENSION FORMULA: Let $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map. Then

This result is also called the **fundamental theorem of linear algebra**.

EXAMPLE: If A is an invertible $n \times n$ matrix, then the dimension of the image is n and that the $\dim(\ker(A)) = 0$.

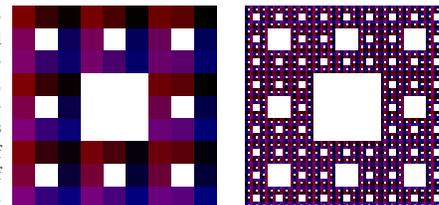
$$\dim(\ker(A)) + \dim(\text{im}(A)) = n$$

PROOF OF THE DIMENSION FORMULA. There are n columns. $\dim(\ker(A))$ is the number of columns without leading 1, $\dim(\text{im}(A))$ is the number of columns with leading 1.

FRactal DIMENSION. Mathematicians study objects with non-integer dimension since the early 20'th century. The topic became fashion in the 80'ies, when mathematicians started to generate fractals on computers. To define fractals, the notion of dimension is extended: define a **s-volume of accuracy** r of a bounded set X in \mathbb{R}^n as the infimum of all $h_{s,r}(X) = \sum |U_j|^s$, where U_j are cubes of length $\leq r$ covering X and $|U_j|$ is the length of U_j . The **s-volume** is then defined as the limit $h_s(X)$ of $h_{s,r}(X) = h_{s,r}(X)$ when $r \rightarrow 0$. The **dimension** is the limiting value s , where $h_s(X)$ jumps from 0 to ∞ . Examples:

- 1) A smooth curve X of length 1 in the plane can be covered with n squares U_j of length $|U_j| = 1/n$ and $h_{s,1/n}(X) = \sum_{j=1}^n (1/n)^s = n(1/n)^s$. If $s < 1$, this converges, if $s > 1$ it diverges for $n \rightarrow \infty$. So $\dim(X) = 1$.
- 2) A square X in space of area 1 can be covered with n^2 cubes U_j of length $|U_j| = 1/n$ and $h_{s,1/n}(X) = \sum_{j=1}^{n^2} (1/n)^s = n^2(1/n)^s$ which converges to 0 for $s < 2$ and diverges for $s > 2$ so that $\dim(X) = 2$.

3) The **Shirpinski carpet** is constructed recursively by dividing a square in 9 equal squares and cutting away the middle one, repeating this procedure with each of the squares etc. At the k 'th step, we need 8^k squares of length $1/3^k$ to cover the carpet. The s -volume $h_{s,1/3^k}(X)$ of accuracy $1/3^k$ is $8^k(1/3^k)^s = 8^k/3^{ks}$, which goes to 0 for $k \rightarrow \infty$ if $3^{ks} < 8^k$ or $s < d = \log(8)/\log(3)$ and diverges if $s > d$. The dimension is $d = \log(8)/\log(3) = 1.893$.



Mathematicians call a fact a "lemma" if it is used to prove a theorem and if does not deserve the be honored by the name "theorem":

COORDINATES

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B-COORDINATES. Given a basis $\vec{v}_1, \dots, \vec{v}_n$, define the matrix $S = \begin{bmatrix} | & \dots & | \\ \vec{v}_1 & \dots & \vec{v}_n \\ | & \dots & | \end{bmatrix}$. It is invertible. If $\vec{x} = c_1\vec{v}_1 + \dots + c_n\vec{v}_n$, then c_i are called the **B-coordinates** of \vec{v} . We write $[\vec{x}]_B = \begin{bmatrix} c_1 \\ \dots \\ c_n \end{bmatrix}$. If $\vec{x} = \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix}$, we have $\vec{x} = S([\vec{x}]_B)$.

B-coordinates of \vec{x} are obtained by applying S^{-1} to the coordinates of the standard basis:

$$[\vec{x}]_B = S^{-1}(\vec{x})$$

EXAMPLE. If $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, then $S = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$. A vector $\vec{v} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$ has the coordinates

$$S^{-1}\vec{v} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 9 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

Indeed, as we can check, $-3\vec{v}_1 + 3\vec{v}_2 = \vec{v}$.

EXAMPLE. Let V be the plane $x + y - z = 1$. Find a basis, in which every vector in the plane has the form $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$. **SOLUTION.** Find a basis, such that two vectors v_1, v_2 are in the plane and such that a third vector v_3 is linearly independent to the first two. Since $(1, 0, 1), (0, 1, 1)$ are points in the plane and $(0, 0, 0)$ is in the plane, we can choose $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ which is perpendicular to the plane.

EXAMPLE. Find the coordinates of $\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ with respect to the basis $B = \{\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$. We have $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $S^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. Therefore $[v]_B = S^{-1}\vec{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Indeed $-1\vec{v}_1 + 3\vec{v}_2 = \vec{v}$.

B-MATRIX. If $B = \{v_1, \dots, v_n\}$ is a basis in \mathbf{R}^n and T is a linear transformation on \mathbf{R}^n , then the **B-matrix** of T is defined as

$$B = \begin{bmatrix} | & \dots & | \\ [T(\vec{v}_1)]_B & \dots & [T(\vec{v}_n)]_B \\ | & \dots & | \end{bmatrix}$$

COORDINATES HISTORY. Cartesian geometry was introduced by Fermat and Descartes (1596-1650) around 1636. It had a large influence on mathematics. Algebraic methods were introduced into geometry. The beginning of the vector concept came only later at the beginning of the 19th Century with the work of Bolzano (1781-1848). The full power of coordinates becomes visible if we allow to chose our coordinate system adapted to the situation. Descartes biography shows how far dedication to the teaching of mathematics can go:
In 1649 Queen Christina of Sweden persuaded Descartes to go to Stockholm. However the Queen wanted to draw tangents at 5 AM. in the morning and Descartes broke the habit of his lifetime of getting up at 11 o'clock. After only a few months in the cold northern climate, walking to the palace at 5 o'clock every morning, he died of pneumonia.



Fermat



Descartes



Christina



Bolzano

CREATIVITY THROUGH LAZINESS? Legend tells that Descartes (1596-1650) introduced coordinates while lying on the bed, watching a fly (around 1630), that Archimedes (285-212 BC) discovered a method to find the volume of bodies while relaxing in the bath and that Newton (1643-1727) discovered Newton's law while lying under an apple tree. Other examples of lazy discoveries are August Kekulé's analysis of the Benzene molecular structure in a dream (a snake biting in its tail revealed the ring structure) or Steven Hawking discovery that black holes can radiate (while shaving). While unclear which of this is actually true (maybe none), there is a pattern:



According David Perkins (at Harvard school of education): "The Eureka effect", many creative breakthroughs have in common: a **long search** without apparent progress, a prevailing moment and **break through**, and finally, a transformation and **realization**. A breakthrough in a lazy moment is typical - but only after long struggle and hard work.

EXAMPLE. Let T be the reflection at the plane $x + 2y + 3z = 0$. Find the transformation matrix B in the basis $\vec{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$. Because $T(\vec{v}_1) = \vec{v}_1 = [\vec{e}_1]_B, T(\vec{v}_2) = \vec{v}_2 = [\vec{e}_2]_B, T(\vec{v}_3) = -\vec{v}_3 = -[\vec{e}_3]_B$, the solution is $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

SIMILARITY. The **B** matrix of A is $B = S^{-1}AS$, where $S = \begin{bmatrix} | & \dots & | \\ \vec{v}_1 & \dots & \vec{v}_n \\ | & \dots & | \end{bmatrix}$. One says B is **similar** to A .

EXAMPLE. If A is similar to B , then $A^2 + A + 1$ is similar to $B^2 + B + 1$. $B = S^{-1}AS, B^2 = S^{-1}B^2S, S^{-1}S = \mathbf{1}, S^{-1}(A^2 + A + 1)S = B^2 + B + 1$.

PROPERTIES OF SIMILARITY. A, B similar and B, C similar, then A, C are similar. If A is similar to B , then B is similar to A .

QUIZZ: If A is a 2×2 matrix and let $S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, What is $S^{-1}AS$?

MAIN IDEA OF CONJUGATION. The transformation S^{-1} maps the coordinates from the standard basis into the coordinates of the new basis. In order to see what a transformation A does in the new coordinates, we first map it back to the old coordinates, apply A and then map it back again to the new coordinates: $B = S^{-1}AS$.

The transformation in standard coordinates. $\vec{v} \xrightarrow{A} A\vec{v}$ $\xleftrightarrow{S^{-1}}$ $\vec{w} = [\vec{v}]_B \xrightarrow{B} B\vec{w}$ The transformation in **B-coordinates.**

QUESTION. Can the matrix A , which belongs to a projection from \mathbf{R}^3 to a plane $x + y + 6z = 0$ be similar to a matrix which is a rotation by 20 degrees around the z axis? No: a non-invertible A can not be similar to an invertible B : if it were, the inverse $A = SBS^{-1}$ would exist: $A^{-1} = SB^{-1}S^{-1}$.

PROBLEM. Find a clever basis for the reflection of a light ray at the line $x + 2y = 0$. $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

SOLUTION. You can achieve $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ with $S = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$.

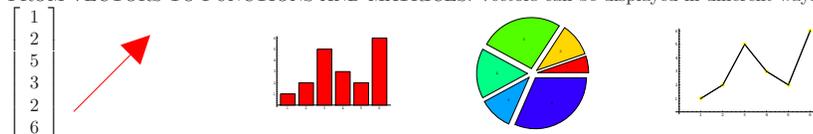
PROBLEM. Are all shears $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ with $a \neq 0$ similar? Yes, use a basis $\vec{v}_1 = a\vec{e}_1$ and $\vec{v}_2 = \vec{e}_2$.

PROBLEM. You know $A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix}$ is similar to $B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ with $S = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$. Find $e^A = \mathbf{1} + A + A^2 + A^3/3! + \dots$ **SOLUTION.** Because $B^k = S^{-1}A^kS$ for every k we have $e^A = Se^B S^{-1}$ and this can be computed, because e^B can be computed easily.

FUNCTION SPACES

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FROM VECTORS TO FUNCTIONS AND MATRICES. Vectors can be displayed in different ways:



The values (i, \vec{v}_i) can be interpreted as the graph of a **function** $f : 1, 2, 3, 4, 5, 6 \rightarrow \mathbf{R}$, where $f(i) = \vec{v}_i$.

Also matrices can be treated as functions, but as a function of two variables. If M is a 8×8 matrix for example, we get a function $f(i, j) = [M]_{ij}$ which assigns to each square of the 8×8 checkerboard a number.

LINEAR SPACES. A space X which contains 0, in which we can add, perform scalar multiplications and where basic laws like commutativity, distributivity and associativity hold, is called a **linear space**.

BASIC EXAMPLE. If A is a set, the space X of all functions from A to \mathbf{R} is a linear space. Here are three important special cases:

EUCLIDEAN SPACE: If $A = \{1, 2, 3, \dots, n\}$, then X is \mathbf{R}^n itself.

FUNCTION SPACE: If A is the real line, then X is a the space of all functions in one variable.

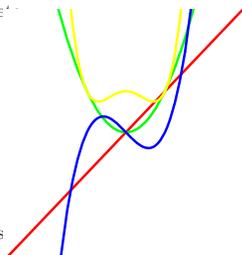
SPACE OF MATRICES: If A is the set

$$\begin{matrix} (1, 1) & (1, 2) & \dots & (1, m) \\ (2, 1) & (2, 2) & \dots & (2, m) \\ \dots & \dots & \dots & \dots \\ (n, 1) & (n, 2) & \dots & (n, m) \end{matrix}$$

Then X is the space of all $n \times m$ matrices.

EXAMPLES.

- The n -dimensional space \mathbf{R}^n .
- linear subspaces of \mathbf{R}^n like the trivial space $\{0\}$, lines or planes ϵ'
- M_n , the space of all square $n \times n$ matrices.
- P_n , the space of all polynomials of degree n .
- The space P of all polynomials.
- C^∞ , the space of all smooth functions on the line
- C^0 , the space of all continuous functions on the line.
- $C^\infty(\mathbf{R}^3, \mathbf{R}^3)$ the space of all smooth vector fields in three dimens
- C^1 , the space of all differentiable functions on the line.
- $C^\infty(\mathbf{R}^3)$ the space of all smooth functions in space.
- L^2 the space of all functions for which $\int_{-\infty}^{\infty} f^2(x) dx < \infty$.



ZERO VECTOR. The function f which is everywhere equal to 0 is called the **zero function**. It plays the role of the zero vector in \mathbf{R}^n . If we add this function to an other function g we get $0 + g = g$.

Careful, the **roots** of a function have nothing to do with the zero function. You should think of the roots of a function like as zero entries of a vector. For the zero vector, all entries have to be zero. For the zero function, all values $f(x)$ are zero.

CHECK: For subsets X of a function space, or for a subset of matrices \mathbf{R}^n , we can check three properties to see whether the space is a linear space:

- if x, y are in X , then $x + y$ is in X .
- If x is in X and λ is a real number, then λx is in X .
- 0 is in X .

WHICH OF THE FOLLOWING ARE LINEAR SPACES?



The space X of all polynomials of the form $f(x) = ax^3 + bx^4 + cx^5$



The space X of all continuous functions on the unit interval $[-1, 1]$ which are zero at -1 and 1. It contains for example the function $f(x) = x^2 - |x|$.



The space X of all smooth periodic functions $f(x+1) = f(x)$. Example $f(x) = \sin(2\pi x) + \cos(6\pi x)$.



The space $X = \sin(x) + C^\infty(\mathbf{R})$ of all smooth functions $f(x) = \sin(x) + g$, where g is a smooth function.



The space X of all trigonometric polynomials $f(x) = a_0 + a_1 \sin(x) + a_2 \sin(2x) + \dots + a_n \sin(nx)$.



The space X of all smooth functions on \mathbf{R} which satisfy $f(1) = 1$. It contains for example $f(x) = 1 + \sin(x) + x$.



The space X of all continuous functions on \mathbf{R} which satisfy $f(2) = 0$ and $f(10) = 0$.



The space X of all smooth functions on \mathbf{R} which satisfy $\lim_{|x| \rightarrow \infty} f(x) = 0$.



The space X of all continuous functions on \mathbf{R} which satisfy $\lim_{|x| \rightarrow \infty} f(x) = 1$.



The space X of all smooth functions on \mathbf{R} of compact support: for every f , there exists an interval I such that $f(x) = 0$ outside that interval.



The space X of all smooth functions on \mathbf{R}^2 .

If you have taken multivariable calculus you might like the following examples:



The space X of all vector fields (P, Q) in the plane, for which the curl $Q_x - P_y$ is zero everywhere.



The space X of all vector fields (P, Q, R) in space, for which the divergence $P_x + Q_y + R_z$ is zero everywhere.



The space X of all vector fields (P, Q) in the plane for which the line integral $\int_C F \cdot dr$ along the unit circle is zero.



The space X of all vector fields (P, Q, R) in space for which the flux through the unit sphere is zero.



The space X of all functions $f(x, y)$ of two variables for which $\int_0^1 \int_0^1 f(x, y) dx dy = 0$.

You have given a matrix $A = \begin{bmatrix} 2 & 1 & 3 & 4 \\ * & * & * & * \end{bmatrix}$, where $*$ can be filled with arbitrary numbers. What are the possible dimensions of the image and kernel?

True or False?

If A is an invertible 3×3 matrix and $\mathcal{B} = \{v_1, v_2, v_3\}$ are the columns of A , then A is the same matrix in the \mathcal{B} coordinates.

Find a basis of the kernel of $A = \begin{bmatrix} 2 & 4 & 6 \\ 2 & 4 & 6 \\ 2 & 4 & 6 \\ 2 & 4 & 6 \end{bmatrix}$.

Find the matrix A of the reflection at the plane $x - y = 0$.

Simplify $B(AB)^{-1}A$.

If A and B are two 5×5 matrices, then the set of vectors \vec{x} satisfying $A\vec{x} = 5B\vec{x}$ is a linear subspace of \mathbf{R}^n .

True or False?

If A is in reduced row echelon form and invertible, then A^{-1} is in reduced row echelon form.

True or False?

There is a matrix A satisfying $A^3 = 0$ and $A^2 \neq 0$.

Find a basis and the dimension of the space X of upper triangular

3×3 matrices $A = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$.

True or False? If A is invertible, then $A - 1$ is invertible.

True or False? The following identity holds for all 2×2 matrices A :

$$(A - 1)^{-1} - (A + 1)^{-1} = 2(A^2 - 1)^{-1}$$

How is the image of A^2 related to the image of A ?

True or False? Assume you have a system of 234 equations with 235 unknowns, then there is a nontrivial kernel.

True or False? The functions

$$f_1(x) = x^4 + 11$$

$$f_2(x) = x^3 + 13$$

$$f_3(x) = x^2 + 15$$

$$f_4(x) = x + 17$$

$$f_5 * x = 19$$

form a basis in the space P_4 of polynomials of degree smaller or equal to 4.

True or False? The following identity $(A^5 - 1)(A - 1)^{-1} = (A^4 + A^3 + A^2 + A + 1)$ holds for all matrices A for which $A - 1$ is invertible.

True or False? If A is invertible and $AB - BA = 0$, then B is invertible.

Which dimensions can the image of the matrix

$$A = \begin{bmatrix} 1 & 1 \\ a & b \\ 2 & 2 \\ 3 & 4 \end{bmatrix}$$

have?

Assume $A = [a \ b \ c]$ and $B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and that not all a, b, c are zero. Which of the matrices AB or BA is always invertible?

The vector $\vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ spans a line in the plane. Find the matrix of the projection onto that line.

You have given matrices A, B, C, D, E in the plane, where A is a shear, B is a reflection, C is a projection, D is a dilation, E is a rotation. There is one of them which is not invertible, which one?

Verify that the identity $(A+B)(A^2-B^2)^{-1} = (A-B)^{-1}$ is true if $AB = BA$ and $A-B$ and $A+B$ are invertible.

You have given matrices A, B, C, D, E in the plane, where A is a shear, B is a reflection, C is a projection, D is a dilation, E is a rotation. There is one matrix X among them which has the property that X^2 is no more of the same type.

True or False? $A\vec{x} = \vec{b}$ can have a unique solution if A is a 4×3 matrix.

True or False? If $A\vec{x} = \vec{b}$ is consistent and has a kernel, then $A\vec{x} = \vec{b}$ has infinitely many solutions.

If A is a 2×2 matrix such that $(A - 1)^2 = (A + 1)^2$. Then $A = 0$.

$A\vec{x} = \vec{b}$ has infinitely many solutions implies that $A^5\vec{x} = \vec{b}$ has infinitely many solutions.

Which of the following two facts is true?

- a) $\text{rref}(A - B) = \text{rref}(A) - \text{rref}(B)$.
- b) $\text{rref}(A) = \text{rref}(\text{rref}(A))$

Find the matrix of a rotation-dilation in the plane, where the rotation has angle $30^\circ = \pi/6$ and the scale factor is 3.

Find a 2×2 matrix A such that $\text{rref}(A) + \mathbf{1}_2 = \text{rref}(B)$ for some matrix B .

Relate $\ker(A^2)$ with $\ker(A)$ where A is a square matrix.

True or False? $A\vec{x} = \vec{b}$ has a unique solution implies that $A^5\vec{x} = \vec{b}$ has a unique solution.

Is $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ in row reduced echelon form?

Relate $\text{im}(A^2)$ with $\text{im}(A)$, where A is a square matrix.

Find the matrix of a rotation-dilation where the rotation has angle $30^\circ = \pi/6$ and the scale factor is 3.

Relate $\ker(A^2)$ with $\ker(A)$ where A is a square matrix.

Is $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ in row reduced echelon form?

Relate $\text{im}(A^2)$ with $\text{im}(A)$, where A is a square matrix.

A square matrix is invertible if and only if $\text{rref}(A)$ is invertible.

If a linear transformation T in the plane satisfies $T^2(x) = x$, then T is a reflection at a line.

If A^{100} is invertible, then A is invertible.

The image and kernel of a matrix are always perpendicular to each other.

The functions $\sin(x)$, $\cos(x)$, e^x span a three dimensional space in $C^\infty(\mathbb{R})$.

The sum of the rank and the nullity is the number of rows of a matrix.

A matrix for which all diagonal entries are 1 is invertible.

There is a shear which is not invertible.

ORTHOGONAL PROJECTIONS

Math 21b, O. Knill

ORTHOGONALITY. Two vectors \vec{v} and \vec{w} are called **orthogonal** if $\vec{v} \cdot \vec{w} = 0$.

Examples. 1) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ -3 \end{bmatrix}$ are orthogonal in \mathbf{R}^2 . 2) \vec{v} and w are both orthogonal to the cross product $\vec{v} \times \vec{w}$ in \mathbf{R}^3 .

\vec{v} is called a **unit vector** if $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = 1$. $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ are called **orthogonal** if they are pairwise orthogonal. They are called **orthonormal** if they are also unit vectors. A basis is called an **orthonormal basis** if it is a basis which is orthonormal. For an orthonormal basis, the matrix $A_{ij} = \vec{v}_i \cdot \vec{v}_j$ is the unit matrix.

FACT: Orthogonal vectors are linearly independent and n orthogonal vectors in \mathbf{R}^n form a basis.

Proof. The dot product of a **linear relation** $a_1\vec{v}_1 + \dots + a_n\vec{v}_n = 0$ with \vec{v}_k gives $a_k\vec{v}_k \cdot \vec{v}_k = a_k\|\vec{v}_k\|^2 = 0$ so that $a_k = 0$. If we have n linear independent vectors in \mathbf{R}^n , they automatically span the space.

ORTHOGONAL COMPLEMENT. A vector $\vec{w} \in \mathbf{R}^n$ is called **orthogonal** to a linear space V , if \vec{w} is orthogonal to every vector $\vec{v} \in V$. The **orthogonal complement** of a linear space V is the set W of all vectors which are orthogonal to V . It forms a linear space because $\vec{v} \cdot \vec{w}_1 = 0, \vec{v} \cdot \vec{w}_2 = 0$ implies $\vec{v} \cdot (\vec{w}_1 + \vec{w}_2) = 0$.

ORTHOGONAL PROJECTION. The **orthogonal projection** onto a linear space V with **orthonormal** basis $\vec{v}_1, \dots, \vec{v}_n$ is the linear map $T(\vec{x}) = \text{proj}_V(x) = (\vec{v}_1 \cdot \vec{x})\vec{v}_1 + \dots + (\vec{v}_n \cdot \vec{x})\vec{v}_n$. The vector $\vec{x} - \text{proj}_V(\vec{x})$ is called the **orthogonal complement** of V . Note that the \vec{v}_i are unit vectors which also have to be orthogonal.

EXAMPLE ENTIRE SPACE: for an orthonormal basis \vec{v}_i of the entire space $\text{proj}_V(x) = \vec{x} = (\vec{v}_1 \cdot \vec{x})\vec{v}_1 + \dots + (\vec{v}_n \cdot \vec{x})\vec{v}_n$. EXAMPLE: if \vec{v} is a unit vector then $\text{proj}_V(x) = (x_1 \cdot v)v$ is the vector projection we know from multi-variable calculus.

PYTHAGORAS: If \vec{x} and \vec{y} are orthogonal, then $\|\vec{x} + \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$. Proof. Expand $(\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y})$.

PROJECTIONS DO NOT INCREASE LENGTH: $\|\text{proj}_V(\vec{x})\| \leq \|\vec{x}\|$. Proof. Use Pythagoras: on $\vec{x} = \text{proj}_V(\vec{x}) + (\vec{x} - \text{proj}_V(\vec{x}))$. If $\|\text{proj}_V(\vec{x})\| = \|\vec{x}\|$, then \vec{x} is in V .

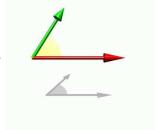
CAUCHY-SCHWARTZ INEQUALITY: $|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|$. Proof: $\vec{x} \cdot \vec{y} = \|\vec{x}\| \|\vec{y}\| \cos(\alpha)$.

If $|\vec{x} \cdot \vec{y}| = \|\vec{x}\| \|\vec{y}\|$, then \vec{x} and \vec{y} are parallel.

TRIANGLE INEQUALITY: $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$. Proof: $(\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) = \|\vec{x}\|^2 + \|\vec{y}\|^2 + 2\vec{x} \cdot \vec{y} \leq \|\vec{x}\|^2 + \|\vec{y}\|^2 + 2\|\vec{x}\| \|\vec{y}\| = (\|\vec{x}\| + \|\vec{y}\|)^2$.

ANGLE. The **angle** between two vectors \vec{x}, \vec{y} is $\alpha = \arccos\left(\frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \|\vec{y}\|}\right)$.

CORRELATION. $\cos(\alpha) = \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \|\vec{y}\|} \in [-1, 1]$ is the **correlation** of \vec{x} and \vec{y} if the vectors \vec{x}, \vec{y} represent data of zero mean.



EXAMPLE. The angle between two orthogonal vectors is 90 degrees or 270 degrees. If \vec{x} and \vec{y} represent data of zero average then $\frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \|\vec{y}\|}$ is called the **statistical correlation** of the data.

QUESTION. Express the fact that \vec{x} is in the kernel of a matrix A using orthogonality.

ANSWER: $A\vec{x} = 0$ means that $\vec{w}_k \cdot \vec{x} = 0$ for every row vector \vec{w}_k of \mathbf{R}^n .

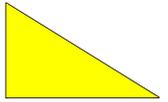
REMARK. We will call later the matrix A^T , obtained by switching rows and columns of A the **transpose** of A . You see already that the image of A^T is orthogonal to the kernel of A .

QUESTION. Find a basis for the orthogonal complement of the linear space V spanned by $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \\ 7 \end{bmatrix}$.

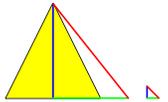
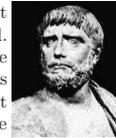
ANSWER: The orthogonality of $\begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix}$ to the two vectors means solving the linear system of equations $x + 2y + 3z + 4u = 0, 4x + 5y + 6z + 7u = 0$. An other way to solve it: the kernel of $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \end{bmatrix}$ is the orthogonal complement of V . This reduces the problem to an older problem.

ON THE RELEVANCE OF ORTHOGONALITY.

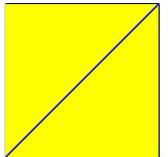
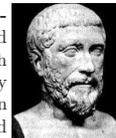
1) From -2800 til -2300 BC, Egyptians used ropes divided into length ratios like 3 : 4 : 5 to build triangles. This allowed them to triangulate areas quite precisely: for example to build irrigation needed because the Nile was reshaping the land constantly or to build the pyramids: for the **great pyramid at Giza** with a base length of 230 meters, the average error on each side is less then 20cm, an error of less then 1/1000. A key to achieve this was **orthogonality**.



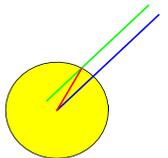
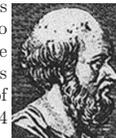
2) During one of Thales (-624 BC to (-548 BC)) journeys to Egypt, he used a geometrical trick to **measure the height** of the great pyramid. He measured the size of the shadow of the pyramid. Using a stick, he found the relation between the length of the stick and the length of its shadow. The same length ratio applies to the pyramid (**orthogonal** triangles). Thales found also that triangles inscribed into a circle and having as the base as the diameter must have a right angle.



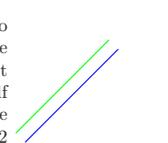
3) The Pythagoreans (-572 until -507) were interested in the discovery that the squares of a lengths of a triangle with two **orthogonal** sides would add up as $a^2 + b^2 = c^2$. They were puzzled in assigning a length to the diagonal of the unit square, which is $\sqrt{2}$. This number is irrational because $\sqrt{2} = p/q$ would imply that $q^2 = 2p^2$. While the prime factorization of q^2 contains an even power of 2, the prime factorization of $2p^2$ contains an odd power of 2.



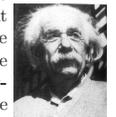
4) Eratosthenes (-274 until 194) realized that while the sun rays were **orthogonal** to the ground in the town of Scene, this did no more do so at the town of Alexandria, where they would hit the ground at 7.2 degrees). Because the distance was about 500 miles and 7.2 is 1/50 of 360 degrees, he measured the circumference of the earth as 25'000 miles - pretty close to the actual value 24'874 miles.



5) Closely related to **orthogonality** is **parallelism**. Mathematicians tried for ages to prove Euclid's parallel axiom using other postulates of Euclid (-325 until -265). These attempts had to fail because there are geometries in which parallel lines always meet (like on the sphere) or geometries, where parallel lines never meet (the Poincaré half plane). Also these geometries can be studied using linear algebra. The geometry on the sphere with **rotations**, the geometry on the half plane uses Möbius transformations, 2×2 matrices with determinant one.



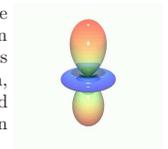
6) The question whether the angles of a right triangle do in reality always add up to 180 degrees became an issue when geometries where discovered, in which the measurement depends on the position in space. Riemannian geometry, founded 150 years ago, is the foundation of **general relativity**, a theory which describes gravity geometrically: the presence of mass bends space-time, where the dot product can depend on space. **Orthogonality** becomes relative too. On a sphere for example, the three angles of a triangle are bigger than 180° . Space is curved.



7) In **probability theory** the notion of **independence** or **decorrelation** is used. For example, when throwing a dice, the number shown by the first dice is independent and decorrelated from the number shown by the second dice. Decorrelation is identical to **orthogonality**, when vectors are associated to the random variables. The **correlation coefficient** between two vectors \vec{v}, \vec{w} is defined as $\vec{v} \cdot \vec{w} / (\|\vec{v}\| \|\vec{w}\|)$. It is the cosine of the angle between these vectors.



8) In **quantum mechanics**, states of atoms are described by functions in a linear space of functions. The states with energy $-E_B/n^2$ (where $E_B = 13.6\text{eV}$ is the Bohr energy) in a hydrogen atom. States in an atom are **orthogonal**. Two states of two different atoms which don't interact are **orthogonal**. One of the challenges in quantum computation, where the computation deals with qubits (=vectors) is that orthogonality is not preserved during the computation (because we don't know all the information). Different states can interact.

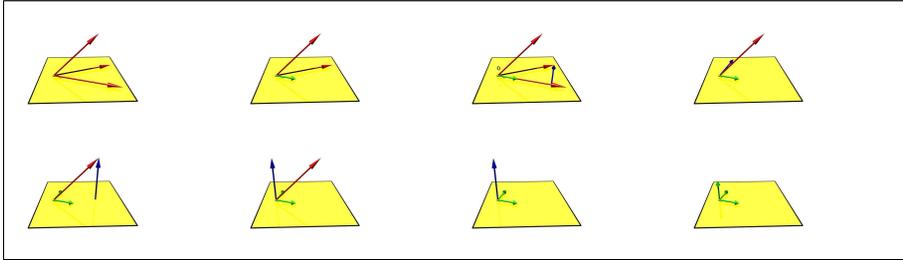


GRAM SCHMIDT AND QR FACTORIZATION

Math 21b, O. Knill

GRAM-SCHMIDT PROCESS.

Let $\vec{v}_1, \dots, \vec{v}_n$ be a basis in V . Let $\vec{u}_1 = \vec{v}_1$ and $\vec{w}_1 = \vec{u}_1 / \|\vec{u}_1\|$. The Gram-Schmidt process recursively constructs from the already constructed orthonormal set $\vec{w}_1, \dots, \vec{w}_{i-1}$ which spans a linear space V_{i-1} the new vector $\vec{u}_i = (\vec{v}_i - \text{proj}_{V_{i-1}}(\vec{v}_i))$ which is orthogonal to V_{i-1} , and then normalizing \vec{u}_i to get $\vec{w}_i = \vec{u}_i / \|\vec{u}_i\|$. Each vector \vec{w}_i is orthonormal to the linear space V_{i-1} . The vectors $\{\vec{w}_1, \dots, \vec{w}_n\}$ form an orthonormal basis in V .



EXAMPLE.

Find an orthonormal basis for $\vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$.

SOLUTION.

- $\vec{w}_1 = \vec{v}_1 / \|\vec{v}_1\| = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.
- $\vec{u}_2 = (\vec{v}_2 - \text{proj}_{V_1}(\vec{v}_2)) = \vec{v}_2 - (\vec{w}_1 \cdot \vec{v}_2)\vec{w}_1 = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$. $\vec{w}_2 = \vec{u}_2 / \|\vec{u}_2\| = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.
- $\vec{u}_3 = (\vec{v}_3 - \text{proj}_{V_2}(\vec{v}_3)) = \vec{v}_3 - (\vec{w}_1 \cdot \vec{v}_3)\vec{w}_1 - (\vec{w}_2 \cdot \vec{v}_3)\vec{w}_2 = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$, $\vec{w}_3 = \vec{u}_3 / \|\vec{u}_3\| = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

QR FACTORIZATION.

The formulas can be written as

$$\vec{v}_1 = \|\vec{w}_1\| \vec{w}_1 = r_{11} \vec{w}_1$$

...

$$\vec{v}_i = (\vec{w}_1 \cdot \vec{v}_i)\vec{w}_1 + \dots + (\vec{w}_{i-1} \cdot \vec{v}_i)\vec{w}_{i-1} + \|\vec{u}_i\| \vec{w}_i = r_{i1}\vec{w}_1 + \dots + r_{ii}\vec{w}_i$$

...

$$\vec{v}_n = (\vec{w}_1 \cdot \vec{v}_n)\vec{w}_1 + \dots + (\vec{w}_{n-1} \cdot \vec{v}_n)\vec{w}_{n-1} + \|\vec{u}_n\| \vec{w}_n = r_{n1}\vec{w}_1 + \dots + r_{nn}\vec{w}_n$$

which means in matrix form

$$A = \begin{bmatrix} | & | & \cdot & | \\ \vec{v}_1 & \dots & \vec{v}_m & \\ | & | & \cdot & | \end{bmatrix} = \begin{bmatrix} | & | & \cdot & | \\ \vec{w}_1 & \dots & \vec{w}_m & \\ | & | & \cdot & | \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & \dots & r_{1m} \\ 0 & r_{22} & \dots & r_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & r_{mm} \end{bmatrix} = QR,$$

where A and Q are $n \times m$ matrices and R is a $m \times m$ matrix.

THE GRAM-SCHMIDT PROCESS PROVES: Any matrix A with linearly independent columns \vec{v}_i can be decomposed as $A = QR$, where Q has orthonormal column vectors and where R is an upper triangular square matrix. The matrix Q has the orthonormal vectors \vec{w}_i in the columns.

BACK TO THE EXAMPLE.

The matrix with the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{bmatrix}$.

so that $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $R = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{bmatrix}$.

$\vec{v}_1 = \|\vec{w}_1\| \vec{w}_1$
 $\vec{v}_2 = (\vec{w}_1 \cdot \vec{v}_2)\vec{w}_1 + \|\vec{u}_2\| \vec{w}_2$
 $\vec{v}_3 = (\vec{w}_1 \cdot \vec{v}_3)\vec{w}_1 + (\vec{w}_2 \cdot \vec{v}_3)\vec{w}_2 + \|\vec{u}_3\| \vec{w}_3,$

PRO MEMORIA.

While building the matrix R we keep track of the vectors u_i during the Gram-Schmidt procedure. At the end you have vectors $\vec{u}_i, \vec{v}_i, \vec{w}_i$ and the matrix R has $\|\vec{u}_i\|$ in the diagonal as well as the dot products $\vec{w}_i \cdot \vec{v}_j$ in the upper right triangle where $i < j$.

PROBLEM. Make the QR decomposition of $A = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$. $\vec{w}_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. $\vec{u}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.
 $\vec{w}_2 = \vec{u}_2$. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = QR$.

WHY do we care to have an orthonormal basis?

- An orthonormal basis looks like the standard basis $\vec{v}_1 = (1, 0, \dots, 0), \dots, \vec{v}_n = (0, 0, \dots, 1)$. Actually, we will see that an orthonormal basis into a standard basis or a mirror of the standard basis.
- The Gram-Schmidt process is tied to the factorization $A = QR$. The later helps to solve linear equations. In physical problems like in astrophysics, the numerical methods to simulate the problems one needs to invert huge matrices in every time step of the evolution. The reason why this is necessary sometimes is to assure the numerical method is stable implicit methods. Inverting $A^{-1} = R^{-1}Q^{-1}$ is easy because R and Q are easy to invert.
- For many physical problems like in quantum mechanics or dynamical systems, matrices are **symmetric** $A^* = A$, where $A_{ij}^* = \bar{A}_{ji}$. For such matrices, there will a natural orthonormal basis.
- The **formula for the projection** onto a linear subspace V simplifies with an orthonormal basis \vec{v}_j in V : $\text{proj}_V(\vec{x}) = (\vec{v}_1 \cdot \vec{x})\vec{w}_1 + \dots + (\vec{w}_n \cdot \vec{x})\vec{w}_n$.
- An orthonormal basis simplifies computations due to the presence of many zeros $\vec{w}_j \cdot \vec{w}_i = 0$. This is especially the case for problems with symmetry.
- The Gram Schmidt process can be used to define and construct classes of classical polynomials, which are important in physics. Examples are Chebyshev polynomials, Laguerre polynomials or Hermite polynomials.
- QR factorization allows fast computation of the determinant, least square solutions $R^{-1}Q^{-1}\vec{b}$ of overdetermined systems $A\vec{x} = \vec{b}$ or finding eigenvalues - all topics which will appear later.

SOME HISTORY.

The recursive formulae of the process were stated by Erhard Schmidt (1876-1959) in 1907. The essence of the formulae were already in a 1883 paper of J.P.Gram in 1883 which Schmidt mentions in a footnote. The process seems already have been used by Laplace (1749-1827) and was also used by Cauchy (1789-1857) in 1836.



Gram

Schmidt

Laplace

Cauchy

ORTHOGONAL MATRICES

Math 21b, O. Knill

TRANSPOSE The **transpose** of a matrix A is the matrix $(A^T)_{ij} = A_{ji}$. If A is a $n \times m$ matrix, then A^T is a $m \times n$ matrix. For square matrices, the transposed matrix is obtained by reflecting the matrix at the diagonal. In general, the rows of A^T are the columns of A .

EXAMPLES The transpose of a vector $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is the row vector $A^T = [1 \ 2 \ 3]$.

The transpose of the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is the matrix $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$.

PROPERTIES.

a) $(AB)^T = B^T A^T$.

b) $\vec{x} \cdot A\vec{y} = A^T \vec{x} \cdot \vec{y}$.

c) $(A^T)^T = A$.

PROOFS.

a) $(AB)_{kl} = \sum_i A_{ki} B_{il}$. $(AB)_{kl}^T = \sum_i A_{li} B_{ik} = A^T B^T$.

b) $\vec{x} \cdot A\vec{y}$ is the matrix product $\vec{x}^T A \vec{y}$. Use a) to see $\vec{x} \cdot A\vec{y} = \vec{x}^T A \vec{y} = (A^T \vec{x})^T \vec{y} = A^T \vec{x} \cdot \vec{y}$.

c) $((A^T)^T)_{ij} = (A^T)_{ji} = A_{ij}$.

ORTHOGONAL MATRIX. A $n \times n$ matrix A is called **orthogonal** if $A^T A = 1$. The corresponding linear transformation is called **orthogonal**.

INVERSE. It is easy to invert an orthogonal matrix because $A^{-1} = A^T$.

EXAMPLES. The rotation matrix $A = \begin{bmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{bmatrix}$ is orthogonal because its column vectors have length 1 and are orthogonal to each other. Indeed: $A^T A = \begin{bmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{bmatrix} \cdot \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. A reflection at a line is an orthogonal transformation because the columns of the matrix A have length 1 and are orthogonal. Indeed: $A^T A = \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix} \cdot \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

PRESERVATION OF LENGTH AND ANGLE. Orthogonal transformations preserve the dot product:

$A\vec{x} \cdot A\vec{y} = \vec{x} \cdot \vec{y}$ Proof. $A\vec{x} \cdot A\vec{y} = A^T A\vec{x} \cdot \vec{y}$ and because of the orthogonality property, this is $\vec{x} \cdot \vec{y}$.

Orthogonal transformations preserve the **length** of vectors as well as the **angles** between them.

Proof. We have $\|A\vec{x}\|^2 = A\vec{x} \cdot A\vec{x} = \vec{x} \cdot \vec{x} = \|\vec{x}\|^2$. Let α be the angle between \vec{x} and \vec{y} and let β denote the angle between $A\vec{x}$ and $A\vec{y}$ and α the angle between \vec{x} and \vec{y} . Using $A\vec{x} \cdot A\vec{y} = \vec{x} \cdot \vec{y}$ we get $\|A\vec{x}\| \|A\vec{y}\| \cos(\beta) = A\vec{x} \cdot A\vec{y} = \vec{x} \cdot \vec{y} = \|\vec{x}\| \|\vec{y}\| \cos(\alpha)$. Because $\|A\vec{x}\| = \|\vec{x}\|$, $\|A\vec{y}\| = \|\vec{y}\|$, this means $\cos(\alpha) = \cos(\beta)$. Because this property holds for all vectors we can rotate \vec{x} in plane V spanned by \vec{x} and \vec{y} by an angle ϕ to get $\cos(\alpha + \phi) = \cos(\beta + \phi)$ for all ϕ . Differentiation with respect to ϕ at $\phi = 0$ shows also $\sin(\alpha) = \sin(\beta)$ so that $\alpha = \beta$.

ORTHOGONAL MATRICES AND BASIS. A linear transformation A is orthogonal if and only if the column vectors of A form an orthonormal basis.

Proof. Look at $A^T A = I_n$. Each entry is a dot product of a column of A with another column of A .

COMPOSITION OF ORTHOGONAL TRANSFORMATIONS. The composition of two orthogonal transformations is orthogonal. The inverse of an orthogonal transformation is orthogonal. Proof. The properties of the transpose give $(AB)^T AB = B^T A^T AB = B^T B = 1$ and $(A^{-1})^T A^{-1} = (A^T)^{-1} A^{-1} = (AA^T)^{-1} = 1_n$.

EXAMPLES.

The composition of two reflections at a line is a rotation.

The composition of two rotations is a rotation.

The composition of a reflections at a plane with a reflection at another plane is a rotation (the axis of rotation is the intersection of the planes).

ORTHOGONAL PROJECTIONS. The orthogonal projection P onto a linear space with orthonormal basis $\vec{v}_1, \dots, \vec{v}_n$ is the matrix $\begin{bmatrix} AA^T \end{bmatrix}$, where A is the matrix with column vectors \vec{v}_i . To see this just translate the formula $P\vec{x} = (\vec{v}_1 \cdot \vec{x})\vec{v}_1 + \dots + (\vec{v}_n \cdot \vec{x})\vec{v}_n$ into the language of matrices: $A^T \vec{x}$ is a vector with components $\vec{b}_i = (\vec{v}_i \cdot \vec{x})$ and $A\vec{b}$ is the sum of the $\vec{v}_i \vec{b}_i$, where \vec{v}_i are the column vectors of A . Orthogonal projections are no orthogonal transformations unless it is the identity!

EXAMPLE. Find the orthogonal projection P from \mathbf{R}^3 to the linear space spanned by $\vec{v}_1 = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} \frac{1}{5}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Solution: $AA^T = \begin{bmatrix} 0 & 1 \\ 3/5 & 0 \\ 4/5 & 0 \end{bmatrix} \begin{bmatrix} 0 & 3/5 & 4/5 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 9/25 & 12/25 \\ 0 & 12/25 & 16/25 \end{bmatrix}$.

WHY ARE ORTHOGONAL TRANSFORMATIONS USEFUL?

- In Physics, Galileo transformations are compositions of translations with orthogonal transformations. The laws of classical mechanics are invariant under such transformations. This is a symmetry.
- Many coordinate transformations are orthogonal transformations. We will see examples when dealing with differential equations.
- In the QR decomposition of a matrix A , the matrix Q is orthogonal. Because $Q^{-1} = Q^t$, this allows to invert A easier.
- Fourier transformations are orthogonal transformations. We will see this transformation later in the course. In application, it is useful in computer graphics (like JPG) and sound compression (like MP3).

WHICH OF THE FOLLOWING MAPS ARE ORTHOGONAL TRANSFORMATIONS?:

Yes	No	Shear in the plane.
Yes	No	Projection in three dimensions onto a plane.
Yes	No	Reflection in two dimensions at the origin.
Yes	No	Reflection in three dimensions at a plane.
Yes	No	Dilation with factor 2.
Yes	No	The Lorenz boost $\vec{x} \mapsto A\vec{x}$ in the plane with $A = \begin{bmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{bmatrix}$
Yes	No	A translation.

CHANGING COORDINATES ON THE EARTH. Problem: what is the matrix which rotates a point on earth with (latitude,longitude)=(a_1, b_1) to a point with (latitude,longitude)=(a_2, b_2)? Solution: The matrix which rotate the point $(0,0)$ to (a, b) a composition of two rotations. The first rotation brings the point into the right latitude, the second brings the point into the right longitude. $R_{a,b} = \begin{bmatrix} \cos(b) & -\sin(b) & 0 \\ \sin(b) & \cos(b) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(a) & 0 & -\sin(a) \\ 0 & 1 & 0 \\ \sin(a) & 0 & \cos(a) \end{bmatrix}$. To bring a point (a_1, b_1) to a point (a_2, b_2) , we form $A = R_{a_2, b_2} R_{a_1, b_1}^{-1}$.



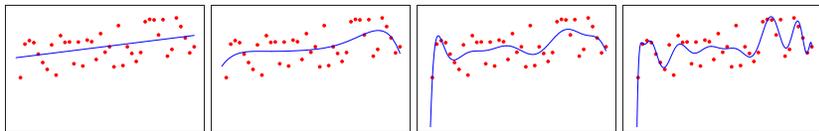
EXAMPLE: With Cambridge (USA): $(a_1, b_1) = (42.366944, 288.893889)\pi/180$ and Zürich (Switzerland): $(a_2, b_2) = (47.377778, 8.551111)\pi/180$, we get the matrix

$$A = \begin{bmatrix} 0.178313 & -0.980176 & -0.0863732 \\ 0.983567 & 0.180074 & -0.0129873 \\ 0.028284 & -0.082638 & 0.996178 \end{bmatrix}$$

LEAST SQUARES AND DATA

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GOAL. The best possible "solution" of an inconsistent linear systems $Ax = b$ is called the **least square solution**. It is the orthogonal projection of b onto the image $\text{im}(A)$ of A . What we know about the kernel and the image of linear transformations helps to understand this situation and leads to an explicit formulas for the least square fit. Why do we care about non-consistent systems? Often we have to solve linear systems of equations with more constraints than variables. An example is when we try to find the best polynomial which passes through a set of points. This problem is called **data fitting**. If we wanted to accommodate all data, the degree of the polynomial would become too large. The fit would look too wiggly. Taking a smaller degree polynomial will not only be more convenient but also give a better picture. Especially important is **regression**, the fitting of data with lines.



The above pictures show 30 data points which are fitted best with polynomials of degree 1, 6, 11 and 16. The first linear fit maybe tells most about the trend of the data.

THE ORTHOGONAL COMPLEMENT OF $\text{im}(A)$. Because a vector is in the kernel of A^T if and only if it is orthogonal to the rows of A^T and so to the columns of A , the kernel of A^T is the orthogonal complement of $\text{im}(A)$: $(\text{im}(A))^\perp = \ker(A^T)$

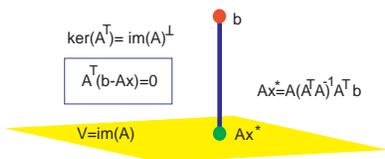
EXAMPLES.

- $A = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. The kernel V of $A^T = [a \ b \ c]$ consists of all vectors satisfying $ax + by + cz = 0$. V is a plane. The orthogonal complement is the image of A which is spanned by the normal vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ to the plane.
- $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. The image of A is spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ the kernel of A^T is spanned by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

ORTHOGONAL PROJECTION. If \vec{b} is a vector and V is a linear subspace, then $\text{proj}_V(\vec{b})$ is the vector closest to \vec{b} on V : given any other vector \vec{v} on V , one can form the triangle $\vec{b}, \vec{v}, \text{proj}_V(\vec{b})$ which has a right angle at $\text{proj}_V(\vec{b})$ and invoke Pythagoras.

THE KERNEL OF $A^T A$. For any $m \times n$ matrix $\ker(A) = \ker(A^T A)$ Proof. \subset is clear. On the other hand $A^T A v = 0$ means that $A v$ is in the kernel of A^T . But since the image of A is orthogonal to the kernel of A^T , we have $A v = 0$, which means \vec{v} is in the kernel of A .

LEAST SQUARE SOLUTION. The **least square solution** of $A\vec{x} = \vec{b}$ is the vector \vec{x}^* such that $A\vec{x}^*$ is closest to \vec{b} from all other vectors $A\vec{x}$. In other words, $A\vec{x}^* = \text{proj}_V(\vec{b})$, where $V = \text{im}(A)$. Because $\vec{b} - A\vec{x}^*$ is in $V^\perp = \text{im}(A)^\perp = \ker(A^T)$, we have $A^T(\vec{b} - A\vec{x}^*) = 0$. The last equation means that \vec{x}^* is a solution of $A^T A \vec{x} = A^T \vec{b}$, the **normal equation** of $A\vec{x} = \vec{b}$. If the kernel of A is trivial, then the kernel of $A^T A$ is trivial and $A^T A$ can be inverted. Therefore $\vec{x}^* = (A^T A)^{-1} A^T \vec{b}$ is the least square solution.



WHY LEAST SQUARES? If \vec{x}^* is the least square solution of $A\vec{x} = \vec{b}$ then $\|A\vec{x}^* - \vec{b}\| \leq \|A\vec{x} - \vec{b}\|$ for all \vec{x} . Proof. $A^T(A\vec{x}^* - \vec{b}) = 0$ means that $A\vec{x}^* - \vec{b}$ is in the kernel of A^T which is orthogonal to $V = \text{im}(A)$. That is $\text{proj}_V(\vec{b}) = A\vec{x}^*$ which is the closest point to \vec{b} on V .

ORTHOGONAL PROJECTION If $\vec{v}_1, \dots, \vec{v}_n$ is a basis in V which is not necessarily orthonormal, then the orthogonal projection is $\vec{x} \mapsto A(A^T A)^{-1} A^T(\vec{x})$ where $A = [\vec{v}_1, \dots, \vec{v}_n]$.

Proof. $\vec{x} = (A^T A)^{-1} A^T \vec{b}$ is the least square solution of $A\vec{x} = \vec{b}$. Therefore $A\vec{x} = A(A^T A)^{-1} A^T \vec{b}$ is the vector in $\text{im}(A)$ closest to \vec{b} .

Special case: If $\vec{w}_1, \dots, \vec{w}_n$ is an orthonormal basis in V , we had seen earlier that AA^T with $A = [\vec{w}_1, \dots, \vec{w}_n]$ is the orthogonal projection onto V (this was just rewriting $A\vec{x} = (\vec{w}_1 \cdot \vec{x})\vec{w}_1 + \dots + (\vec{w}_n \cdot \vec{x})\vec{w}_n$ in matrix form.) This follows from the above formula because $A^T A = I$ in that case.

EXAMPLE Let $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 1 \end{bmatrix}$. The orthogonal projection onto $V = \text{im}(A)$ is $\vec{b} \mapsto A(A^T A)^{-1} A^T \vec{b}$. We have

$$A^T A = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix} \text{ and } A(A^T A)^{-1} A^T = \begin{bmatrix} 1/5 & 2/5 & 0 \\ 2/5 & 4/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For example, the projection of $\vec{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is $\vec{x}^* = \begin{bmatrix} 2/5 \\ 4/5 \\ 0 \end{bmatrix}$ and the distance to \vec{b} is $1/\sqrt{5}$. The point \vec{x}^* is the point on V which is closest to \vec{b} .

Remember the formula for the distance of \vec{b} to a plane V with normal vector \vec{n} ? It was $d = |\vec{n} \cdot \vec{b}|/|\vec{n}|$. In our case, we can take $\vec{n} = [-2, 1, 0]$ and get the distance $1/\sqrt{5}$. Let's check: the distance of \vec{x}^* and \vec{b} is $\| (2/5, -1/5, 0) \| = 1/\sqrt{5}$.

EXAMPLE. Let $A = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$. Problem: find the matrix of the orthogonal projection onto the image of A .

The image of A is a one-dimensional line spanned by the vector $\vec{v} = (1, 2, 0, 1)$. We calculate $A^T A = 6$. Then

$$A(A^T A)^{-1} A^T = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} [1 \ 2 \ 0 \ 1] / 6 = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} / 6$$

DATA FIT. Find a quadratic polynomial $p(t) = at^2 + bt + c$ which best fits the four data points $(-1, 8), (0, 8), (1, 4), (2, 16)$.

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 8 \\ 8 \\ 4 \\ 16 \end{bmatrix} \quad A^T A = \begin{bmatrix} 18 & 8 & 6 \\ 8 & 6 & 2 \\ 6 & 2 & 4 \end{bmatrix} \text{ and } \vec{x}^* = (A^T A)^{-1} A^T \vec{b} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$$

Software packages like Mathematica have already built in the facility to fit numerical data:

```

DataPoints = {{(-1,8), (0,8), (1,4), (2,16)}}
f=Function[y,Fit[DataPoints, {1,x,x^2},x] /. x->y];
Show[ {ListPlot[DataPoints], Plot[f[t], {t,-1.2,2}]}];
Series[f[x], {x,0,2}]
                    
```

The series expansion of f showed that indeed, $f(t) = 5 - t + 3t^2$ is indeed best quadratic fit. Actually, Mathematica does the same to find the fit then what we do: **"Solving" an inconsistent system of linear equations as best as possible.**

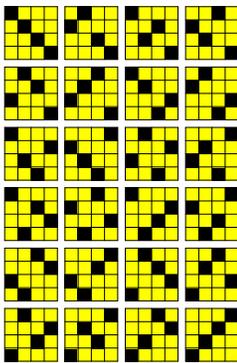
PROBLEM: Prove $\text{im}(A) = \text{im}(AA^T)$.

SOLUTION. The image of AA^T is contained in the image of A because we can write $\vec{v} = AA^T \vec{x}$ as $\vec{v} = A\vec{y}$ with $\vec{y} = A^T \vec{x}$. On the other hand, if \vec{v} is in the image of A , then $\vec{v} = A\vec{x}$. If $\vec{x} = \vec{y} + \vec{z}$, where \vec{y} in the kernel of A and \vec{z} orthogonal to the kernel of A , then $A\vec{x} = A\vec{z}$. Because \vec{z} is orthogonal to the kernel of A , it is in the image of A^T . Therefore, $\vec{z} = A^T \vec{u}$ and $\vec{v} = A\vec{z} = AA^T \vec{u}$ is in the image of AA^T .

DETERMINANTS I

Math 21b, O. Knill

PERMUTATIONS. A **permutation** of $\{1, 2, \dots, n\}$ is a rearrangement of $\{1, 2, \dots, n\}$. There are $n! = n \cdot (n-1) \cdot \dots \cdot 1$ different permutations of $\{1, 2, \dots, n\}$: fixing the position of first element leaves $(n-1)!$ possibilities to permute the rest.



EXAMPLE. There are 6 permutations of $\{1, 2, 3\}$: $(1, 2, 3)$, $(1, 3, 2)$, $(2, 1, 3)$, $(2, 3, 1)$, $(3, 1, 2)$, $(3, 2, 1)$.

PATTERNS AND SIGN. The matrix A with zeros everywhere except $A_{i,\pi(i)} = 1$ is called a permutation matrix or the **pattern** of π . An **upcrossing** is a pair $k < l$ such that $\pi(k) < \pi(l)$. The **sign** of a permutation π is defined as $\text{sign}(\pi)$ where $\text{sign}(\pi)$ is the number of upcrossings in the pattern of π . that is the number pairs of black squares, where the upper square is to the right.

EXAMPLES. $\text{sign}(1, 2) = 0$, $\text{sign}(2, 1) = 1$. $\text{sign}(1, 2, 3) = \text{sign}(3, 2, 1) = \text{sign}(2, 3, 1) = 1$. $\text{sign}(1, 3, 2) = \text{sign}(3, 2, 1) = \text{sign}(2, 1, 3) = -1$.

DETERMINANT The **determinant** of a $n \times n$ matrix $A = a_{ij}$ is defined as the sum

$$\sum_{\pi} \text{sign}(\pi) a_{1\pi(1)} a_{2\pi(2)} \cdots a_{n\pi(n)}$$

where π is a permutation of $\{1, 2, \dots, n\}$.

2 × 2 CASE. The determinant of $A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is $ad - bc$. There are two permutations of $(1, 2)$. The identity permutation $(1, 2)$ gives $a_{11}a_{22}$, the permutation $(2, 1)$ gives $a_{21}a_{12}$. If you have seen some multi-variable calculus, you know that $\det(A)$ is the area of the parallelogram spanned by the column vectors of A . The two vectors form a basis if and only if $\det(A) \neq 0$.

3 × 3 CASE. The determinant of $A = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ is $a_{ei} + b_{fg} + c_{dh} - c_{eg} - f_{ha} - b_{di}$ corresponding to the 6 permutations of $(1, 2, 3)$. Geometrically, $\det(A)$ is the volume of the parallelepiped spanned by the column vectors of A . The three vectors form a basis if and only if $\det(A) \neq 0$.

EXAMPLE DIAGONAL AND TRIANGULAR MATRICES. The determinant of a diagonal or triangular matrix is the product of the diagonal elements.

EXAMPLE PERMUTATION MATRICES. The determinant of a matrix which has everywhere zeros except $a_{i\pi(j)} = 1$ is just the sign $\text{sign}(\pi)$ of the permutation.

THE LAPLACE EXPANSION. To compute the determinant of a $n \times n$ matrices $A = a_{ij}$. Choose a column i . For each entry a_{ji} in that column, take the $(n-1) \times (n-1)$ matrix A_{ji} which does not contain the i 'th column and j 'th row. The determinant of A_{ji} is called a **minor**. One gets

$$\det(A) = (-1)^{i+1} a_{i1} \det(A_{i1}) + \cdots + (-1)^{i+n} a_{in} \det(A_{in}) = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det(A_{ij})$$

This Laplace expansion is just a convenient arrangement of the permutations: listing all permutations of the form $(1, *, \dots, *)$ of n elements is the same then listing all permutations of $(2, *, \dots, *)$ of $(n-1)$ elements etc.

TRIANGULAR AND DIAGONAL MATRICES. The determinant of a **diagonal** or **triangular** matrix is the product of its diagonal elements.

Example: $\det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 5 & 0 & 0 \\ 2 & 3 & 4 & 0 \\ 1 & 1 & 2 & 1 \end{pmatrix} = 20$.

PARTITIONED MATRICES.

The determinant of a **partitioned matrix** $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is the product $\det(A)\det(B)$.

Example $\det \begin{pmatrix} 3 & 4 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 2 & 2 \end{pmatrix} = 2 \cdot 12 = 24$.

LINEARITY OF THE DETERMINANT. If the columns of A and B are the same except for the i 'th column,

$$\det([v_1, \dots, v, \dots, v_n]) + \det([v_1, \dots, w, \dots, v_n]) = \det([v_1, \dots, v+w, \dots, v_n])$$

In general, one has $\det([v_1, \dots, kv, \dots, v_n]) = k \det([v_1, \dots, v, \dots, v_n])$. The same identities hold for rows and follow directly from the original definition of the determinant.

ROW REDUCED ECHELON FORM. Determining $\text{rref}(A)$ also determines $\det(A)$.

If A is a matrix and $\lambda_1, \dots, \lambda_k$ are the factors which are used to scale different rows and s is the total number of times, two rows were switched, then $\det(A) = (-1)^s \alpha_1 \cdots \lambda_k \det(\text{rref}(A))$.

INVERTIBILITY. Because of the last formula: A $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$.

PROBLEM. Find the determinant of $A = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 1 & 2 & 4 & 5 \\ 0 & 7 & 2 & 9 \\ 0 & 0 & 6 & 4 \end{bmatrix}$.

SOLUTION. Three row transpositions give $B = \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 7 & 2 & 9 \\ 0 & 0 & 6 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ a matrix which has determinant 84. Therefore $\det(A) = (-1)^3 \det(B) = -84$.

PROPERTIES OF DETERMINANTS. (see also next lecture)

$$\det(AB) = \det(A)\det(B)$$

$$\det(SAS^{-1}) = \det(A)$$

$$\det(\lambda A) = \lambda^n \det(A)$$

$$\det(A^{-1}) = \det(A)^{-1}$$

$$\det(A^T) = \det(A)$$

$$\det(-A) = (-1)^n \det(A)$$

If B is obtained from A by switching two rows, then $\det(B) = -\det(A)$. If B is obtained by adding an other row to a given row, then this does not change the value of the determinant.

PROOF OF $\det(AB) = \det(A)\det(B)$, one brings the $n \times n$ matrix $[A|B]$ into row reduced echelon form. Similar than the augmented matrix $[A|b]$ was brought into the form $[1|A^{-1}b]$, we end up with $[1|A^{-1}AB] = [1|B]$. By looking at the $n \times n$ matrix to the left during Gauss-Jordan elimination, the determinant has changed by a factor $\det(A)$. We end up with a matrix B which has determinant $\det(B)$. Therefore, $\det(AB) = \det(A)\det(B)$. PROOF OF $\det(A^T) = \det(A)$. The transpose of a pattern is a pattern with the same signature.

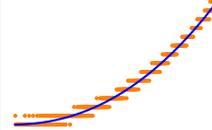
PROBLEM. Determine $\det(A^{100})$, where A is the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 16 \end{bmatrix}$.

SOLUTION. $\det(A) = 10$, $\det(A^{100}) = (\det(A))^{100} = 10^{100} = 1 \cdot \text{gogool}$. This name as well as the gogoolplex = $10^{10^{100}}$ are official. They are huge numbers: the mass of the universe for example is 10^{52}kg and $1/10^{10^{51}}$ is the chance to find yourself on Mars by quantum fluctuations. (R.E. Crandall, Scient. Amer., Feb. 1997).

ORTHOGONAL MATRICES. Because $Q^T Q = 1$, we have $\det(Q)^2 = 1$ and so $|\det(Q)| = 1$. Rotations have determinant 1, reflections can have determinant -1 .

QR DECOMPOSITION. If $A = QR$, then $\det(A) = \det(Q)\det(R)$. The determinant of Q is ± 1 , the determinant of R is the product of the diagonal elements of R .

HOW FAST CAN WE COMPUTE THE DETERMINANT?.



The cost to find the determinant is the same as for the Gauss-Jordan elimination. A measurements of the time needed for mathematica to compute a determinant of a random $n \times n$ matrix. The matrix size ranged from $n=1$ to $n=300$. We see a cubic fit of these data.

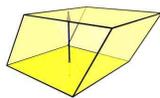
WHY DO WE CARE ABOUT DETERMINANTS?

- check invertibility of matrices
- allow to define orientation in any dimensions
- have geometric interpretation as volume
- appear in change of variable formulas in higher dimensional integration.
- explicit algebraic expressions for inverting a matrix
- proposed alternative concepts are unnatural, hard to teach and harder to understand
- as a natural functional on matrices it appears in formulas in particle or statistical physics
- determinants are fun

DETERMINANTS II

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DETERMINANT AND VOLUME. If A is a $n \times n$ matrix, then $|\det(A)|$ is the volume of the n -dimensional parallelepiped E_n spanned by the n column vectors v_j of A .



Proof. Use the QR decomposition $A = QR$, where Q is orthogonal and R is upper triangular. From $QQ^T = I$, we get $1 = \det(Q)\det(Q^T) = \det(Q)^2$ see that $|\det(Q)| = 1$. Therefore, $\det(A) = \pm \det(R)$. The determinant of R is the product of the $\|u_i\| = \|v_i - \text{proj}_{V_{j-1}} v_i\|$ which was the distance from v_i to V_{j-1} . The volume $\text{vol}(E_j)$ of a j -dimensional parallelepiped E_j with base E_{j-1} in V_{j-1} and height $\|u_j\|$ is $\text{vol}(E_{j-1})\|u_j\|$. Inductively $\text{vol}(E_j) = \|u_j\|\text{vol}(E_{j-1})$ and therefore $\text{vol}(E_n) = \prod_{j=1}^n \|u_j\| = \det(R)$.

The volume of a k dimensional parallelepiped defined by the vectors v_1, \dots, v_k is $\sqrt{|\det(A^T A)|}$.

Proof. $Q^T Q = I_n$ gives $A^T A = (QR)^T(QR) = R^T Q^T Q R = R^T R$. So, $\det(R^T R) = \det(R)^2 = (\prod_{j=1}^k \|u_j\|)^2$. (Note that A is a $n \times k$ matrix and that $A^T A = R^T R$ and R are $k \times k$ matrices.)

ORIENTATION. Determinants allow to **define** the orientation of n vectors in n -dimensional space. This is "handy" because there is no "right hand rule" in hyperspace ... To do so, define the matrix A with column vectors v_j and define the orientation as the sign of $\det(A)$. In three dimensions, this agrees with the right hand rule: if v_1 is the thumb, v_2 is the pointing finger and v_3 is the middle finger, then their orientation is positive.

$x_i \det(A) =$

CRAMER'S RULE. This is an explicit formula for the solution of $A\vec{x} = \vec{b}$. If A_i denotes the matrix, where the column \vec{v}_i of A is replaced by \vec{b} , then

$$x_i = \det(A_i) / \det(A)$$

Proof. $\det(A_i) = \det([v_1, \dots, b, \dots, v_n]) = \det([v_1, \dots, (Ax), \dots, v_n]) = \det([v_1, \dots, \sum_i x_i v_i, \dots, v_n]) = x_i \det([v_1, \dots, v_i, \dots, v_n]) = x_i \det(A)$

EXAMPLE. Solve the system $5x+3y = 8, 8x+5y = 2$ using Cramers rule. This linear system with $A = \begin{bmatrix} 5 & 3 \\ 8 & 5 \end{bmatrix}$ and $b = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$. We get $x = \det \begin{bmatrix} 8 & 3 \\ 2 & 5 \end{bmatrix} = 34y = \det \begin{bmatrix} 5 & 8 \\ 8 & 2 \end{bmatrix} = -54$.

GABRIEL CRAMER. (1704-1752), born in Geneva, Switzerland, he worked on geometry and analysis. Cramer used the rule named after him in a book "Introduction à l'analyse des lignes courbes algébrique", where he solved like this a system of equations with 5 unknowns. According to a short biography of Cramer by J.J O'Connor and E F Robertson, the rule had however been used already before by other mathematicians. Solving systems with Cramers formulas is slower than by Gaussian elimination. The rule is still important. For example, if A or b depends on a parameter t , and we want to see how x depends on the parameter t one can find explicit formulas for $(d/dt)x_i(t)$.

THE INVERSE OF A MATRIX. Because the columns of A^{-1} are solutions of $A\vec{x} = \vec{e}_i$, where \vec{e}_j are basis vectors, Cramers rule together with the Laplace expansion gives the formula:

$$[A^{-1}]_{ij} = (-1)^{i+j} \det(A_{ji}) / \det(A)$$

$B_{ij} = (-1)^{i+j} \det(A_{ji})$ is called the **classical adjoint** of A . **Note** the change $ij \rightarrow ji$. **Don't** confuse the classical adjoint with the **transpose** A^T which is sometimes also called the **adjoint**.

EXAMPLE. $A = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 2 & 4 \\ 6 & 0 & 7 \end{bmatrix}$ has $\det(A) = -17$ and we get $A^{-1} = \begin{bmatrix} 14 & -21 & 10 \\ -11 & 8 & -3 \\ -12 & 18 & -11 \end{bmatrix} / (-17)$:

$B_{11} = (-1)^{1+1} \det \begin{bmatrix} 2 & 4 \\ 0 & 7 \end{bmatrix} = 14$. $B_{12} = (-1)^{1+2} \det \begin{bmatrix} 3 & 1 \\ 0 & 7 \end{bmatrix} = -21$. $B_{13} = (-1)^{1+3} \det \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} = 10$.

$B_{21} = (-1)^{2+1} \det \begin{bmatrix} 5 & 4 \\ 6 & 7 \end{bmatrix} = -11$. $B_{22} = (-1)^{2+2} \det \begin{bmatrix} 2 & 1 \\ 6 & 7 \end{bmatrix} = 8$. $B_{23} = (-1)^{2+3} \det \begin{bmatrix} 2 & 1 \\ 5 & 4 \end{bmatrix} = -3$.

$B_{31} = (-1)^{3+1} \det \begin{bmatrix} 5 & 2 \\ 6 & 0 \end{bmatrix} = -12$. $B_{32} = (-1)^{3+2} \det \begin{bmatrix} 2 & 3 \\ 6 & 0 \end{bmatrix} = 18$. $B_{33} = (-1)^{3+3} \det \begin{bmatrix} 2 & 3 \\ 5 & 2 \end{bmatrix} = -11$.

THE ART OF CALCULATING DETERMINANTS. When confronted with a matrix, it is good to go through a checklist of methods to crack the determinant. Often, there are different possibilities to solve the problem, in many cases the solution is particularly simple using one method.

- Is it a 2×2 or 3×3 matrix?
- Do you see duplicated columns or rows?
- Is it an upper or lower triangular matrix?
- Can you row reduce to a triangular case?
- Is it a partitioned matrix?
- Are there only a few nonzero patterns?
- Is it a product like $\det(A^{1000}) = \det(A)^{1000}$?
- Try Laplace expansion with some row or column?
- Is the matrix known to be noninvertible.
- Later: Can you see the eigenvalues of A ?

EXAMPLES.

1) $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 6 & 8 & 10 \\ 5 & 5 & 5 & 5 & 4 \\ 1 & 3 & 2 & 7 & 4 \\ 3 & 2 & 8 & 4 & 9 \end{bmatrix}$ Try row reduction.

2) $A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$ Laplace expansion.

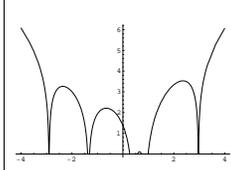
3) $A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 4 & 2 \end{bmatrix}$ Partitioned matrix.

4) $A = \begin{bmatrix} 1 & 6 & 10 & 1 & 15 \\ 2 & 8 & 17 & 1 & 29 \\ 0 & 0 & 3 & 8 & 12 \\ 0 & 0 & 0 & 4 & 9 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$ Make it triangular.

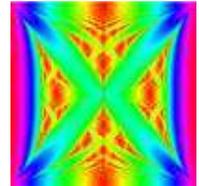
APPLICATION HOFSTADTER BUTTERFLY. In solid state physics, one is interested in the function $f(E) = \det(L - EI_n)$, where

$$L = \begin{bmatrix} \lambda \cos(\alpha) & 1 & 0 & \dots & 0 & 1 \\ 1 & \lambda \cos(2\alpha) & 1 & \dots & \dots & 0 \\ 0 & 1 & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & 1 & 0 \\ 0 & \dots & \dots & 1 & \lambda \cos((n-1)\alpha) & 1 \\ 1 & 0 & \dots & 0 & 1 & \lambda \cos(n\alpha) \end{bmatrix}$$

describes an electron in a periodic crystal, E is the energy and $\alpha = 2\pi/n$. The electron can move as a Bloch wave whenever the determinant is negative. These intervals form the **spectrum** of the quantum mechanical system. A physicist is interested in the rate of change of $f(E)$ or its dependence on λ when E is fixed. .



The graph to the left shows the function $E \mapsto \log(|\det(L - EI_n)|)$ in the case $\lambda = 2$ and $n = 5$. In the energy intervals, where this function is zero, the electron can move, otherwise the crystal is an insulator. The picture to the right shows the spectrum of the crystal depending on α . It is called the "Hofstadter butterfly" made popular in the book "Gödel, Escher Bach" by Douglas Hofstadter.



EIGENVALUES & DYNAMICAL SYSTEMS

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EIGENVALUES AND EIGENVECTORS.

A nonzero vector v is called an **eigenvector** of A with **eigenvalue** λ if $Av = \lambda v$.

EXAMPLES.

- \vec{v} is an eigenvector to the eigenvalue 0 if \vec{v} is in the kernel of A .
- A rotation in space has an eigenvalue 1.
- If A is a diagonal matrix with diagonal elements a_i , then \vec{e}_i is an eigenvector with eigenvalue a_i .
- A shear A in the direction v has an eigenvector \vec{v} .
- Projections have eigenvalues 1 or 0.
- Reflections have eigenvalues 1 or -1 .
- A rotation in the plane by an angle 30 degrees has no real eigenvector. (there are eigenvectors but they are complex).

LINEAR DYNAMICAL SYSTEMS.

Iterating a linear map $x \mapsto Ax$ is called a **discrete dynamical system**. One wants to understand what happens with $x_1 = Ax, x_2 = AAx = A^2x, x_3 = AAAx = A^3x, \dots$

EXAMPLE 1: $x \mapsto ax$ or $x_{n+1} = ax_n$ has the solution $x_n = a^n x_0$. For example, $1.03^{20} \cdot 1000 = 1806.11$ is the balance on a bank account which had 1000 dollars 20 years ago and if the interest rate was constant 3 percent.

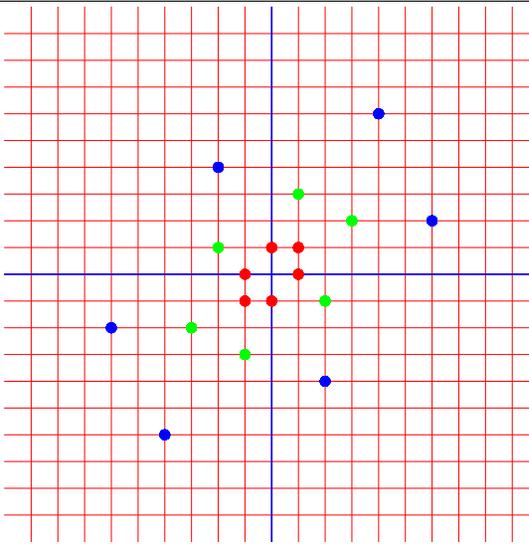
EXAMPLE 2: $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. $A\vec{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $A^2\vec{v} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$. $A^3\vec{v} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$. $A^4\vec{v} = \begin{bmatrix} 9 \\ 1 \end{bmatrix}$ etc.

EXAMPLE 3: If \vec{v} is an eigenvector with eigenvalue λ , then $A\vec{v} = \lambda\vec{v}, A^2\vec{v} = A(A\vec{v}) = A\lambda\vec{v} = \lambda A\vec{v} = \lambda^2\vec{v}$ and more generally $A^n\vec{v} = \lambda^n\vec{v}$.

RECURSION: If a scalar quantity u_{n+1} does not only depend on u_n but also on u_{n-1} we can write $(x_n, y_n) = (u_n, u_{n-1})$ and get a linear map because x_{n+1}, y_{n+1} depend in a linear way on x_n, y_n .

A RECURSION PROBLEM. A linear recursion problem which appears in quantum mechanics is $u_{n+1} + u_{n-1} = Eu_n$ and $u_0 = 0, u_1 = 1$. Because $\begin{bmatrix} E & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_n \\ u_{n-1} \end{bmatrix} = \begin{bmatrix} u_{n+1} \\ u_n \end{bmatrix}$. The recursion is done by iterating the matrix A . Lets take $E = 1$:
 $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$ $A^2 = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$
 $A^3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. We see that A^3 is a reflection at the origin which has the eigenvalue $\lambda = -1$ and A^6 is the identity. Every initial vector is mapped after 6 iterations back to its original starting point.

If the E parameter is changed, the dynamics also changes. For $E = 3$ for example, most initial points will escape to infinity similar as in the next example. Indeed, for $E = 3$, there is an eigenvector $\vec{v} = (3 + \sqrt{5})/2$ to the eigenvalue $\lambda = (3 + \sqrt{5})/2$ and $A^n\vec{v} = \lambda^n\vec{v}$ escapes to ∞ .



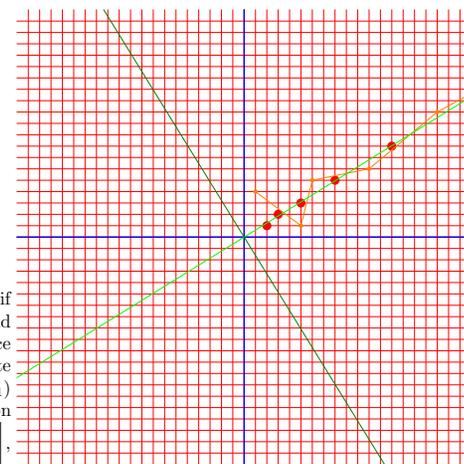
THE FIBONNACCI RECURSION:

In the third section of Liber abacci, published in 1202 **Leonardo Fibonacci** (1170-1250) writes:



A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?

Mathematically, how does u_n grow, if $u_{n+1} = u_n + u_{n-1}$? We can assume $u_0 = 1$ and $u_1 = 2$ to match Leonardo's example. The sequence is $(1, 2, 3, 5, 8, 13, 21, \dots)$. As before we can write this recursion using vectors $(x_n, y_n) = (u_n, u_{n-1})$ starting with $(1, 2)$. The matrix A to this recursion is $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Iterating gives $A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $A^2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = A \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$.



SOLUTION KNOWING EIGENSYSTEM. If $A\vec{v}_1 = \lambda_1\vec{v}_1, A\vec{v}_2 = \lambda_2\vec{v}_2$ and $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2$, we have an explicit solution $A^n\vec{v} = c_1\lambda_1^n\vec{v}_1 + c_2\lambda_2^n\vec{v}_2$. This motivates to find good methods to compute eigenvalues and eigenvectors.

EVOLUTION OF QUANTITIES. Example could be market systems, population quantities of different species, or ingredient quantities in a chemical reaction. A linear description might not always be a good model but it has the advantage that we can solve the system explicitly. Eigenvectors will provide the key to do so.

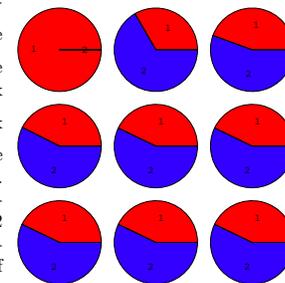
MARKOV MATRICES. A matrix with nonzero entries for which the sum of the columns entries add up to 1 is called a **Markov matrix**.

Markov Matrices have an eigenvalue 1.

Proof. The eigenvalues of A and A^T are the same because they have the same characteristic polynomial. The matrix A^T has an eigenvector $[1, 1, 1, 1]^T$. An example is the matrix

$$A = \begin{bmatrix} 1/2 & 1/3 & 1/4 \\ 1/4 & 1/3 & 1/3 \\ 1/4 & 1/3 & 5/12 \end{bmatrix}$$

MARKOV PROCESS EXAMPLE: The percentage of people using Apple OS or the Linux OS is represented by a vector $\begin{bmatrix} m \\ l \end{bmatrix}$. Each cycle 2/3 of Mac OS users switch to Linux and 1/3 stays. Also lets assume that 1/2 of the Linux OS users switch to apple and 1/2 stay. The matrix $P = \begin{bmatrix} 1/3 & 1/2 \\ 2/3 & 1/2 \end{bmatrix}$ is a **Markov matrix**. What ratio of Apple/Linux users do we have after things settle to an equilibrium? We can simulate this with a dice: start in a state like $M = (1, 0)$ (all users have Macs). If the dice shows 3,4,5 or 6, a user in that group switch to Linux, otherwise stays in the M camp. Throw a dice for each user in L. If 1,2 or 3 shows up, the user switches to M. The matrix P has an eigenvector $(3/7, 4/7)$ which belongs to the eigenvalue 1. The interpretation of $P\vec{v} = \vec{v}$ is that with this split up, there is no change in average.



COMPUTING EIGENVALUES

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THE TRACE. The **trace** of a matrix A is the sum of its diagonal elements.

EXAMPLES. The trace of $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$ is $1 + 4 + 8 = 13$. The trace of a skew symmetric matrix A is zero because there are zeros in the diagonal. The trace of I_n is n .

CHARACTERISTIC POLYNOMIAL. The polynomial $f_A(\lambda) = \det(A - \lambda I_n)$ is called the **characteristic polynomial** of A .

EXAMPLE. The characteristic polynomial of the matrix A above is $p_A(\lambda) = -\lambda^3 + 13\lambda^2 + 15\lambda$.

The eigenvalues of A are the roots of the characteristic polynomial $f_A(\lambda)$.

Proof. If λ is an eigenvalue of A with eigenfunction \vec{v} , then $A - \lambda$ has \vec{v} in the kernel and $A - \lambda$ is not invertible so that $f_A(\lambda) = \det(A - \lambda) = 0$.

The polynomial has the form

$$f_A(\lambda) = (-\lambda)^n + \text{tr}(A)(-\lambda)^{n-1} + \dots + \det(A)$$

THE 2x2 CASE. The characteristic polynomial of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $f_A(\lambda) = \lambda^2 - (a + d)/2\lambda + (ad - bc)$. The eigenvalues are $\lambda_{\pm} = T/2 \pm \sqrt{(T/2)^2 - D}$, where T is the trace and D is the determinant. In order that this is real, we must have $(T/2)^2 \geq D$.

EXAMPLE. The characteristic polynomial of $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ is $\lambda^2 - 3\lambda + 2$ which has the roots 1, 2: $f_A(\lambda) = (1 - \lambda)(2 - \lambda)$.

THE FIBONNACCI RABBITS. The Fibonacci's recursion $u_{n+1} = u_n + u_{n-1}$ defines the growth of the rabbit population. We have seen that it can be rewritten as $\begin{bmatrix} u_{n+1} \\ u_n \end{bmatrix} = A \begin{bmatrix} u_n \\ u_{n-1} \end{bmatrix}$ with $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. The roots of the characteristic polynomial $f_A(x) = \lambda^2 - \lambda - 1$ are $(\sqrt{5} + 1)/2, (\sqrt{5} - 1)/2$.

ALGEBRAIC MULTIPLICITY. If $f_A(\lambda) = (\lambda - \lambda_0)^k g(\lambda)$, where $g(\lambda_0) \neq 0$ then λ is said to be an eigenvalue of **algebraic multiplicity** k .

EXAMPLE: $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ has the eigenvalue $\lambda = 1$ with algebraic multiplicity 2 and the eigenvalue $\lambda = 2$ with algebraic multiplicity 1.

HOW TO COMPUTE EIGENVECTORS? Because $(A - \lambda)v = 0$, the vector v is in the kernel of $A - \lambda$. We know how to compute the kernel.

EXAMPLE FIBONNACCI. The kernel of $A - \lambda I_2 = \begin{bmatrix} 1 - \lambda_{\pm} & 1 \\ 1 & 1 - \lambda_{\pm} \end{bmatrix}$ is spanned by $\vec{v}_+ = [(1 + \sqrt{5})/2, 1]^T$ and $\vec{v}_- = [(1 - \sqrt{5})/2, 1]^T$. They form a basis \mathcal{B} .

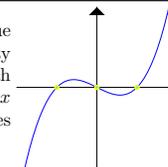
SOLUTION OF FIBONNACCI. To obtain a formula for $A^n \vec{v}$ with $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we form $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} / \sqrt{5}$.

Now, $\begin{bmatrix} u_{n+1} \\ u_n \end{bmatrix} = A^n \vec{v} = A^n (\vec{v}_+ / \sqrt{5} - \vec{v}_- / \sqrt{5}) = A^n \vec{v}_+ / \sqrt{5} - A^n \vec{v}_- / \sqrt{5} = \lambda_+^n \vec{v}_+ / \sqrt{5} - \lambda_-^n \vec{v}_- / \sqrt{5}$. We see that $u_n = [(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n] / \sqrt{5}$.

ROOTS OF POLYNOMIALS.

For polynomials of degree 3 and 4 there exist explicit formulas in terms of radicals. As Galois (1811-1832) and Abel (1802-1829) have shown, it is not possible for equations of degree 5 or higher. Still, one can compute the roots numerically.

REAL SOLUTIONS. A $(2n + 1) \times (2n + 1)$ matrix A always has a real eigenvalue because the characteristic polynomial $p(x) = x^5 + \dots + \det(A)$ has the property that $p(x)$ goes to $\pm\infty$ for $x \rightarrow \pm\infty$. Because there exist values a, b for which $p(a) < 0$ and $p(b) > 0$, by the intermediate value theorem, there exists a real x with $p(x) = 0$. Application: A rotation in 11 dimensional space has all eigenvalues $|\lambda| = 1$. The real eigenvalue must have an eigenvalue 1 or -1 .



EIGENVALUES OF TRANSPOSE. We know that the characteristic polynomials of A and the transpose A^T agree because $\det(B) = \det(B^T)$ for any matrix. Therefore A and A^T have the same eigenvalues.

APPLICATION: MARKOV MATRICES. A matrix A for which each column sums up to 1 is called a **Markov matrix**.

The transpose of a Markov matrix has the eigenvector $\begin{bmatrix} 1 \\ 1 \\ \dots \\ 1 \end{bmatrix}$ with eigenvalue 1. Therefore:

A Markov matrix has an eigenvector \vec{v} to the eigenvalue 1.

This vector \vec{v} defines an equilibrium point of the Markov process.

EXAMPLE. If $A = \begin{bmatrix} 1/3 & 1/2 \\ 2/3 & 1/2 \end{bmatrix}$. Then $[3/7, 4/7]$ is the equilibrium eigenvector to the eigenvalue 1.

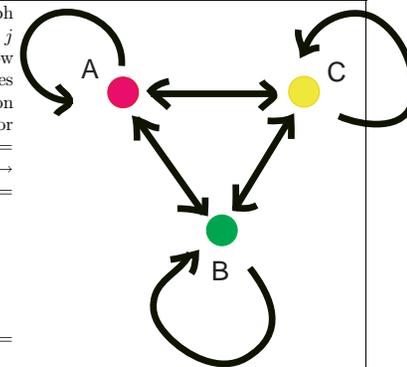
BRETSCHERS HOMETOWN. Problem 28 in the book deals with a Markov problem in Andelfingen the hometown of Bretscher, where people shop in two shops. (Andelfingen is a beautiful village at the Thur river in the middle of a "wine country"). Initially all shop in shop W . After a new shop opens, every week 20 percent switch to the other shop M . Missing something at the new place, every week, 10 percent switch back. This leads to a Markov matrix $A = \begin{bmatrix} 8/10 & 1/10 \\ 2/10 & 9/10 \end{bmatrix}$. After some time, things will settle down and we will have certain percentage shopping in W and other percentage shopping in M . This is the equilibrium.



MARKOV PROCESS IN PROBABILITY. Assume we have a graph like a network and at each node i , the probability to go from i to j in the next step is $[A]_{ij}$, where A_{ij} is a Markov matrix. We know from the above result that there is an eigenvector \vec{p} which satisfies $A\vec{p} = \vec{p}$. It can be normalized that $\sum_i p_i = 1$. The interpretation is that p_i is the probability that the walker is on the node p . For example, on a triangle, we can have the probabilities: $P(A \rightarrow B) = 1/2, P(A \rightarrow C) = 1/4, P(A \rightarrow A) = 1/4, P(B \rightarrow A) = 1/3, P(B \rightarrow B) = 1/6, P(B \rightarrow C) = 1/2, P(C \rightarrow A) = 1/2, P(C \rightarrow B) = 1/3, P(C \rightarrow C) = 1/6$. The corresponding matrix is

$$A = \begin{bmatrix} 1/4 & 1/3 & 1/2 \\ 1/2 & 1/6 & 1/3 \\ 1/4 & 1/2 & 1/6 \end{bmatrix}$$

In this case, the eigenvector to the eigenvalue 1 is $p = [38/107, 36/107, 33/107]^T$.



CALCULATING EIGENVECTORS

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NOTATION. We often just write 1 instead of the identity matrix I_n or λ instead of λI_n .

COMPUTING EIGENVALUES. Recall: because $A - \lambda$ has \vec{v} in the kernel if λ is an eigenvalue the characteristic polynomial $f_A(\lambda) = \det(A - \lambda) = 0$ has eigenvalues as roots.

2 x 2 CASE. Recall: The characteristic polynomial of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $f_A(\lambda) = \lambda^2 - (a+d)/2\lambda + (ad-bc)$. The eigenvalues are $\lambda_{\pm} = T/2 \pm \sqrt{(T/2)^2 - D}$, where $T = a + d$ is the trace and $D = ad - bc$ is the determinant of A . If $(T/2)^2 \geq D$, then the eigenvalues are real. Away from that parabola in the (T, D) space, there are two different eigenvalues. The map A contracts volume for $|D| < 1$.

NUMBER OF ROOTS. Recall: There are examples with no real eigenvalue (i.e. rotations). By inspecting the graphs of the polynomials, one can deduce that $n \times n$ matrices with odd n always have a real eigenvalue. Also $n \times n$ matrixes with even n and a negative determinant always have a real eigenvalue.

IF ALL ROOTS ARE REAL. $f_A(\lambda) = (-\lambda)^n + \text{tr}(A)(-\lambda)^{n-1} + \dots + \det(A) = (\lambda_1 - \lambda) \dots (\lambda_n - \lambda)$, we see that $\lambda_1 + \dots + \lambda_n = \text{trace}(A)$ and $\lambda_1 \cdot \lambda_2 \cdot \dots \cdot \lambda_n = \det(A)$.

HOW TO COMPUTE EIGENVECTORS? Because $(\lambda - A)\vec{v} = 0$, the vector \vec{v} is in the kernel of $\lambda - A$.

EIGENVECTORS of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are \vec{v}_{\pm} with eigenvalue λ_{\pm} .
 If $c \neq 0$, then the eigenvectors to λ_{\pm} are $\begin{bmatrix} \lambda_{\pm} - d \\ c \end{bmatrix}$.
 If $c = d = 0$, then $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are eigenvectors.
 If $b \neq 0$, then the eigenvectors to λ_{\pm} are $\begin{bmatrix} b \\ \lambda_{\pm} - d \end{bmatrix}$.

ALGEBRAIC MULTIPLICITY. If $f_A(\lambda) = (\lambda - \lambda_0)^k g(\lambda)$, where $g(\lambda_0) \neq 0$, then f has **algebraic multiplicity** k . If A is similar to an upper triangular matrix B , then it is the number of times that λ_0 occurs in the diagonal of B .

EXAMPLE: $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ has the eigenvalue $\lambda = 1$ with algebraic multiplicity 2 and eigenvalue 2 with algebraic multiplicity 1.

GEOMETRIC MULTIPLICITY. The dimension of the eigenspace E_{λ} of an eigenvalue λ is called the **geometric multiplicity** of λ .

EXAMPLE: the matrix of a shear is $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. It has the eigenvalue 1 with algebraic multiplicity 2. The kernel of $A - 1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and the geometric multiplicity is 1.

EXAMPLE: The matrix $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ has eigenvalue 1 with algebraic multiplicity 2 and the eigenvalue 0 with multiplicity 1. Eigenvectors to the eigenvalue $\lambda = 1$ are in the kernel of $A - 1$ which is the kernel of $\begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and spanned by $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. The geometric multiplicity is 1.

RELATION BETWEEN ALGEBRAIC AND GEOMETRIC MULTIPLICITY. The geometric multiplicity is smaller or equal than the algebraic multiplicity.

PRO MEMORIAM. You can remember this with an analogy. The **geometric mean** \sqrt{ab} of two numbers is smaller or equal to the **algebraic mean** $(a + b)/2$.

EXAMPLE. What are the algebraic and geometric multiplicities of $A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$?

SOLUTION. The algebraic multiplicity of the eigenvalue 2 is 5. To get the kernel of $A - 2$, one solves the system of equations $x_4 = x_3 = x_2 = x_1 = 0$ so that the geometric multiplicity of the eigenvalues 2 is 1.

CASE: ALL EIGENVALUES ARE DIFFERENT.

If all eigenvalues are different, then all eigenvectors are linearly independent and all geometric and algebraic multiplicities are 1.

PROOF. Let λ_i be an eigenvalue different from 0 and assume the eigenvectors are linearly dependent. We have $v_i = \sum_{j \neq i} a_j v_j$ and $\lambda_i v_i = A v_i = A(\sum_{j \neq i} a_j v_j) = \sum_{j \neq i} a_j \lambda_j v_j$ so that $v_i = \sum_{j \neq i} b_j v_j$ with $b_j = a_j \lambda_j / \lambda_i$. If the eigenvalues are different, then $a_j \neq b_j$ and by subtracting $v_i = \sum_{j \neq i} a_j v_j$ from $v_i = \sum_{j \neq i} b_j v_j$, we get $0 = \sum_{j \neq i} (b_j - a_j) v_j = 0$. Now $(n - 1)$ eigenvectors of the n eigenvectors are linearly dependent. Use induction.

CONSEQUENCE. If all eigenvalues of a $n \times n$ matrix A are different, there is an **eigenbasis**, a basis consisting of eigenvectors.

EXAMPLES. 1) $A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$ has eigenvalues 1, 3 to the eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.
 2) $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ has an eigenvalue 3 with eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ but no other eigenvector. We do not have a basis.
 3) For $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, every vector is an eigenvector. The standard basis is an eigenbasis.

TRICKS: Wonder where teachers take examples? Here are some tricks:

1) If the matrix is upper triangular or lower triangular one can read off the eigenvalues at the diagonal. The eigenvalues can be computed fast because row reduction is easy.

2) For 2×2 matrices, one can immediately write down the eigenvalues and eigenvectors:

The eigenvalues of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are $\lambda_{\pm} = \frac{\text{tr}(A) \pm \sqrt{(\text{tr}(A))^2 - 4\det(A)}}{2}$

The eigenvectors in the case $c \neq 0$ are

$$v_{\pm} = \begin{bmatrix} \lambda_{\pm} - d \\ c \end{bmatrix}.$$

If $b \neq 0$, we have the eigenvectors

$$v_{\pm} = \begin{bmatrix} b \\ \lambda_{\pm} - a \end{bmatrix}$$

If both b and c are zero, then the standard basis is the eigenbasis.

3) How do we construct 2x2 matrices which have integer eigenvectors and integer eigenvalues? Just take an integer matrix for which the row vectors have the same sum. Then this sum is an eigenvalue to the eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The other eigenvalue can be obtained by noticing that the trace of the matrix is the sum of the eigenvalues. For example, the matrix $\begin{bmatrix} 6 & 7 \\ 2 & 11 \end{bmatrix}$ has the eigenvalue 13 and because the sum of the eigenvalues is 18 a second eigenvalue 5.

4) If you see a partitioned matrix

$$C = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

then the union of the eigenvalues of A and B are the eigenvalues of C . If v is an eigenvector of A , then $\begin{bmatrix} v \\ 0 \end{bmatrix}$ is an eigenvector of C . If w is an eigenvector of B , then $\begin{bmatrix} 0 \\ w \end{bmatrix}$ is an eigenvector of C .

DIAGONALIZATION

Math 21b, O.Knill

SUMMARY. A $n \times n$ matrix, $A\vec{v} = \lambda\vec{v}$ with eigenvalue λ and eigenvector \vec{v} . The eigenvalues are the roots of the characteristic polynomial $f_A(\lambda) = \det(\lambda - A) = \lambda^n - \text{tr}(A)\lambda^{n-1} + \dots + (-1)^n \det(A)$. The eigenvectors to the eigenvalue λ are in $\ker(\lambda - A)$. The number of times, an eigenvalue λ occurs in the full list of n roots of $f_A(\lambda)$ is called algebraic multiplicity. It is bigger or equal than the geometric multiplicity: $\dim(\ker(\lambda - A))$.

EXAMPLE. The eigenvalues of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are $\lambda_{\pm} = T/2 \pm \sqrt{T^2/4 - D}$, where $T = a + d$ is the trace and $D = ad - bc$ is the determinant of A . If $c \neq 0$, the eigenvectors are $v_{\pm} = \begin{bmatrix} \lambda_{\pm} - d \\ c \end{bmatrix}$.

If $c = 0$, then a, d are eigenvalues to the eigenvectors $\begin{bmatrix} a \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -b \\ a - d \end{bmatrix}$. If $a = d$, then the second eigenvector is parallel to the first and the geometric multiplicity of the eigenvalue $a = d$ is 1.

EIGENBASIS. If there are n eigenvectors of a $n \times n$ matrix, then A these vectors form a basis called **eigenbasis**. If A has n different eigenvalues, then A has an eigenbasis, consisting of eigenvectors of A .

DIAGONALIZATION. How does the matrix A look in an eigenbasis? If S is the matrix with the eigenvectors as columns, then $[B = S^{-1}AS]$ is diagonal. We have $S\vec{e}_i = \vec{v}_i$ and $AS\vec{e}_i = \lambda_i\vec{v}_i$ we know $S^{-1}AS\vec{e}_i = \lambda_i\vec{e}_i$. Therefore, B is diagonal with diagonal entries λ_i .

EXAMPLE. $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ has the eigenvalues $\lambda_1 = 2 + \sqrt{3}$ with eigenvector $\vec{v}_1 = [\sqrt{3}, 1]$ and the eigenvalues $\lambda_2 = 2 - \sqrt{3}$ with eigenvector $\vec{v}_2 = [-\sqrt{3}, 1]$. Form $S = \begin{bmatrix} \sqrt{3} & -\sqrt{3} \\ 1 & 1 \end{bmatrix}$ and check $S^{-1}AS = D$ is diagonal.

APPLICATION: FUNCTIONAL CALCULUS. Let A be the matrix in the above example. What is $A^{100} + A^{37} - 1$? The trick is to diagonalize A : $B = S^{-1}AS$, then $B^k = S^{-1}A^kS$ and We can compute $A^{100} + A^{37} - 1 = S(B^{100} + B^{37} - 1)S^{-1}$.

APPLICATION: SOLVING LINEAR SYSTEMS. $x(t+1) = Ax(t)$ has the solution $x(n) = A^n x(0)$. To compute A^n , we diagonalize A and get $x(n) = SB^n S^{-1}x(0)$. This is an explicit formula.

SIMILAR MATRICES HAVE THE SAME EIGENVALUES.

One can see this in two ways:

1) If $B = S^{-1}AS$ and \vec{v} is an eigenvector of B to the eigenvalue λ , then $S\vec{v}$ is an eigenvector of A to the eigenvalue λ .

2) From $\det(S^{-1}AS) = \det(A)$, we know that the characteristic polynomials $f_B(\lambda) = \det(\lambda - B) = \det(\lambda - S^{-1}AS) = \det(S^{-1}(\lambda - AS)) = \det((\lambda - A)S) = \det(\lambda - A) = f_A(\lambda)$ are the same.

CONSEQUENCES.

1) Because the characteristic polynomials of similar matrices agree, the trace $\text{tr}(A)$ of similar matrices agrees.

2) The trace is the sum of the eigenvalues of A . Compare the trace of A with the trace of the diagonalize matrix.

THE CAYLEY HAMILTON THEOREM. If A is diagonalizable, then $f_A(A) = 0$.

PROOF. The **DIAGONALIZATION** $B = S^{-1}AS$ has the eigenvalues in the diagonal. So $f_A(B)$, which contains $f_A(\lambda_i)$ in the diagonal is zero. From $f_A(B) = 0$ we get $Sf_A(B)S^{-1} = f_A(A) = 0$.

The theorem holds for all matrices: the coefficients of a general matrix can be changed a tiny bit so that all eigenvalues are different. For any such perturbations one has $f_A(A) = 0$. Because the coefficients of $f_A(A)$ depend continuously on A , they are zero in general.

CRITERIA FOR SIMILARITY.

- If A and B have the same characteristic polynomial and diagonalizable, then they are similar.
- If A and B have a different determinant or trace, they are not similar.
- If A has an eigenvalue which is not an eigenvalue of B , then they are not similar.

AN IMPORTANT THEOREM.

If all eigenvalues of a matrix A are different, then the matrix A is diagonalizable.

WHY DO WE WANT TO DIAGONALIZE?

1) **FUNCTIONS OF MATRICES.** If $p(x) = 1 + x + x^2 + x^3/3! + x^4/4!$ be a polynomial and A is a matrix, then $p(A) = 1 + A + A^2/2! + A^3/3! + A^4/4!$ is a matrix. If $B = S^{-1}AS$ is diagonal with diagonal entries λ_i , then $p(B)$ is diagonal with diagonal entries $p(\lambda_i)$. And $p(A) = S p(B)S^{-1}$. This speeds up the calculation because matrix multiplication costs much. The matrix $p(A)$ can be written down with three matrix multiplications, because $p(B)$ is diagonal.

2) **SOLVING LINEAR DIFFERENTIAL EQUATIONS.** A differential equation $\dot{\vec{v}} = A\vec{v}$ is solved by $\vec{v}(t) = e^{At}\vec{v}(0)$, where $e^{At} = 1 + At + A^2t^2/2! + A^3t^3/3! + \dots$ (Differentiate this sum with respect to t to get $Ae^{At}\vec{v}(0) = A\vec{v}(t)$.) If we write this in an eigenbasis of A , then

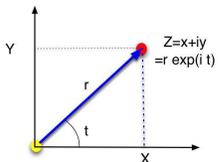
$\vec{y}(t) = e^{Bt}\vec{y}(0)$ with the diagonal matrix $B = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$. In other words, we have

then explicit solutions $y_j(t) = e^{\lambda_j t} y_j(0)$. Linear differential equations later in this course. It is important motivation.

3) **MARKOV PROCESSES.** Complicated systems can be modeled by putting probabilities on each possible event and computing the probabilities that an event switches to any other event. This defines a transition matrix. Such a matrix always has an eigenvalue 1. The corresponding eigenvector is the stable probability distribution on the states. If we want to understand, how fast things settle to this equilibrium, we need to know the other eigenvalues and eigenvectors.

COMPLEX EIGENVALUES

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NOTATION. Complex numbers are written as $z = x + iy = r \exp(i\phi) = r \cos(\phi) + ir \sin(\phi)$. The real number $r = |z|$ is called the **absolute value** of z , the value ϕ is the **argument** and denoted by $\arg(z)$. Complex numbers contain the **real numbers** $z = x + i0$ as a subset. One writes $\text{Re}(z) = x$ and $\text{Im}(z) = y$ if $z = x + iy$.

ARITHMETIC. Complex numbers are added like vectors: $x + iy + u + iv = (x + u) + i(y + v)$ and multiplied as $z * w = (x + iy)(u + iv) = xu - yv + i(yu - xv)$. If $z \neq 0$, one can divide $1/z = 1/(x + iy) = (x - iy)/(x^2 + y^2)$.

ABSOLUTE VALUE AND ARGUMENT. The absolute value $|z| = \sqrt{x^2 + y^2}$ satisfies $|zw| = |z| |w|$. The argument satisfies $\arg(zw) = \arg(z) + \arg(w)$. These are direct consequences of the polar representation $z = r \exp(i\phi)$, $w = s \exp(i\psi)$, $zw = rs \exp(i(\phi + \psi))$.

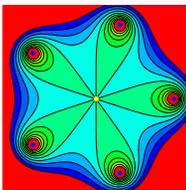
GEOMETRIC INTERPRETATION. If $z = x + iy$ is written as a vector $\begin{bmatrix} x \\ y \end{bmatrix}$, then multiplication with an other complex number w is a **dilation-rotation**: a scaling by $|w|$ and a rotation by $\arg(w)$.

THE DE MOIVRE FORMULA. $z^n = \exp(in\phi) = \cos(n\phi) + i \sin(n\phi) = (\cos(\phi) + i \sin(\phi))^n$ follows directly from $z = \exp(i\phi)$ but it is magic: it leads for example to formulas like $\cos(3\phi) = \cos(\phi)^3 - 3 \cos(\phi) \sin^2(\phi)$ which would be more difficult to come by using geometrical or power series arguments. This formula is useful for example in integration problems like $\int \cos(x)^3 dx$, which can be solved by using the above deMoivre formula.

THE UNIT CIRCLE. Complex numbers of length 1 have the form $z = \exp(i\phi)$ and are located on the **unit circle**. The characteristic polynomial $f_A(\lambda) =$

$$\lambda^5 - 1 \text{ of the matrix } \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ has all roots on the unit circle. The}$$

roots $\exp(2\pi ki/5)$, for $k = 0, \dots, 4$ lye on the unit circle.



THE LOGARITHM. $\log(z)$ is defined for $z \neq 0$ as $\log|z| + i\arg(z)$. For example, $\log(2i) = \log(2) + i\pi/2$. Riddle: what is i^i ? ($i^i = e^{i \log(i)} = e^{i i \pi/2} = e^{-\pi/2}$). The logarithm is not defined at 0 and the imaginary part is define only up to 2π . For example, both $i\pi/2$ and $5i\pi/2$ are equal to $\log(i)$.

HISTORY. The struggle with $\sqrt{-1}$ is historically quite interesting. Nagging questions appeared for example when trying to find closed solutions for roots of polynomials. Cardano (1501-1576) was one of the mathematicians who at least considered complex numbers but called them arithmetic subtleties which were "as refined as useless". With Bombelli (1526-1573), complex numbers found some practical use. Descartes (1596-1650) called roots of negative numbers "imaginary".

Although the fundamental theorem of algebra (below) was still not proved in the 18th century, and complex numbers were not fully understood, the square root of minus one $\sqrt{-1}$ was used more and more. Euler (1707-1783) made the observation that $\exp(ix) = \cos x + i \sin x$ which has as a special case the **magic formula** $e^{i\pi} + 1 = 0$ which relate the constants 0, 1, π , e in one equation.

For decades, many mathematicians still thought complex numbers were a **waste of time**. Others used complex numbers extensively in their work. In 1620, Girard suggested that an equation may have as many roots as its degree in 1620. Leibniz (1646-1716) spent quite a bit of time trying to apply the laws of algebra to complex numbers. He and Johann Bernoulli used imaginary numbers as integration aids. Lambert used complex numbers for map projections, d'Alembert used them in hydrodynamics, while Euler, D'Alembert and Lagrange used them in their incorrect proofs of the fundamental theorem of algebra. Euler write first the symbol i for $\sqrt{-1}$.

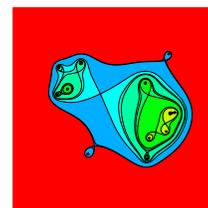
Gauss published the first correct proof of the fundamental theorem of algebra in his doctoral thesis, but still claimed in 1825 that **the true metaphysics of the square root of -1 is elusive** as late as 1825. By 1831 Gauss overcame his uncertainty about complex numbers and published his work on the geometric representation of complex numbers as points in the plane. In 1797, a Norwegian Caspar Wessel (1745-1818) and in 1806 a Swiss clerk named Jean Robert Argand (1768-1822) (who stated the theorem the first time for polynomials with complex coefficients) did similar work. But these efforts went unnoticed. William Rowan Hamilton (1805-1865) (who would also discover the quaternions while walking over a bridge) expressed in 1833 complex numbers as vectors.

Complex numbers continued to develop to **complex function theory** or **chaos theory**, a branch of dynamical systems theory. Complex numbers are helpful in geometry in number theory or in quantum mechanics. Once believed fictitious they are now most "natural numbers" and the "natural numbers" themselves are in fact the most "complex". A philosopher who asks "does $\sqrt{-1}$ really exist?" might be shown the representation of $x + iy$ as $\begin{bmatrix} x & -y \\ y & x \end{bmatrix}$. When adding or multiplying such dilation-rotation matrices, they behave like complex numbers: for example $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ plays the role of i .

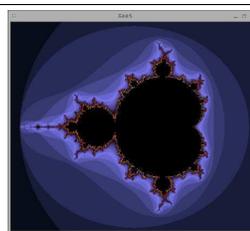
FUNDAMENTAL THEOREM OF ALGEBRA. (Gauss 1799) A polynomial of degree n has exactly n roots.

CONSEQUENCE: A $n \times n$ MATRIX HAS n EIGENVALUES. The characteristic polynomial $f_A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0$ satisfies $f_A(\lambda) = (\lambda - \lambda_1) \dots (\lambda - \lambda_n)$, where λ_i are the roots of f .

TRACE AND DETERMINANT. Comparing $f_A(\lambda) = (\lambda - \lambda_1) \dots (\lambda - \lambda_n)$ with $\lambda^n - \text{tr}(A) \lambda + \dots + (-1)^n \det(A)$ gives $\text{tr}(A) = \lambda_1 + \dots + \lambda_n$, $\det(A) = \lambda_1 \dots \lambda_n$.



COMPLEX FUNCTIONS. The characteristic polynomial is an example of a function f from \mathbb{C} to \mathbb{C} . The graph of this function would live in $\mathbb{C} \times \mathbb{C}$ which corresponds to a four dimensional real space. One can visualize the function however with the real-valued function $z \mapsto |f(z)|$. The figure to the left shows the contour lines of such a function $z \mapsto |f(z)|$, where f is a polynomial.



ITERATION OF POLYNOMIALS. A topic which is off this course (it would be a course by itself) is the iteration of polynomials like $f_c(z) = z^2 + c$. The set of parameter values c for which the iterates $f_c(0), f_c^2(0) = f_c(f_c(0)), \dots, f_c^n(0)$ stay bounded is called the **Mandelbrot set**. It is the fractal black region in the picture to the left. The now already dusty object appears everywhere, from photoshop plugins to decorations. In Mathematica, you can compute the set very quickly (see <http://www.math.harvard.edu/computing/math/mandelbrot.m>).

COMPLEX NUMBERS IN MATHEMATICA OR MAPLE. In both computer algebra systems, the letter I is used for $i = \sqrt{-1}$. In Maple, you can ask $\log(1 + I)$, in Mathematica, this would be $\text{Log}[1 + I]$. Eigenvalues or eigenvectors of a matrix will in general involve complex numbers. For example, in Mathematica, $\text{Eigenvalues}[A]$ gives the eigenvalues of a matrix A and $\text{Eigensystem}[A]$ gives the eigenvalues and the corresponding eigenvectors.

EIGENVALUES AND EIGENVECTORS OF A ROTATION. The rotation matrix $A = \begin{bmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{bmatrix}$ has the characteristic polynomial $\lambda^2 - 2 \cos(\phi) \lambda + 1$. The eigenvalues are $\cos(\phi) \pm \sqrt{\cos^2(\phi) - 1} = \cos(\phi) \pm i \sin(\phi) = \exp(\pm i\phi)$. The eigenvector to $\lambda_1 = \exp(i\phi)$ is $v_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$ and the eigenvector to the eigenvector $\lambda_2 = \exp(-i\phi)$ is $v_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$.

FINANCE: INTEREST RATE.

$$A(x) = 1.02x$$

BIOLOGY: POPULATION GROWTH.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Fibonacci recursion $x_{n+1} = x_n + x_{n-1}$.

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$$

Lila Bush growth.

PHYSICS: SOLID STATE PHYSICS.

$$A = \begin{bmatrix} E & -1 \\ 1 & 0 \end{bmatrix}$$

solves time independent Schrödinger equation $Hu = Eu$ which is $u_{n+1} + u_{n-1} = Eu_n$.

STATISTICS: MARKOV PROCESSES.

$$A = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}$$

Regular transition matrix for Wipf/Migros competition.

$$A = \begin{bmatrix} 0.6 & 0.1 & 0.5 \\ 0.2 & 0.7 & 0.1 \\ 0.2 & 0.2 & 0.4 \end{bmatrix}$$

MCI/ATT/Sprint customers.

ECOLOGY: DIFFUSION.

$$A = \begin{bmatrix} 0.7 & 0 & 0 \\ 0.1 & 0.6 & 0 \\ 0 & 0.2 & 0.8 \end{bmatrix}$$

Silvaplana, Sils, St Moritz lakes.

GEOMETRY: TRANSFORMATIONS.

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

shear.

$$A = 1/2 \begin{bmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{bmatrix}$$

rotation dilation.

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

projection

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

reflection

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

scaling transformation

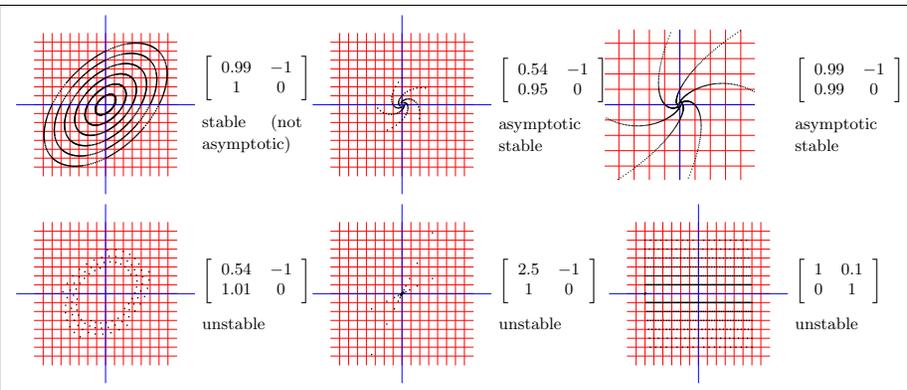
STABILITY

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LINEAR DYNAMICAL SYSTEM. A linear map $x \mapsto Ax$ defines a **dynamical system**. Iterating the linear map produces an **orbit** $x_0, x_1 = Ax, x_2 = A^2 = AAx, \dots$. The vector $x_n = A^n x_0$ describes the situation of the system at **time** n .

Where does x_n go, when time evolves? Can one describe what happens asymptotically when time n goes to infinity?

In the case of the Fibonacci sequence x_n which gives the number of rabbits in a rabbit population at time n , the population grows exponentially. Such a behavior is called **unstable**. On the other hand, if A is a rotation, then $A^n \vec{v}$ stays bounded which is a type of **stability**. If A is a dilation with a dilation factor < 1 , then $A^n \vec{v} \rightarrow 0$ for all \vec{v} , a thing which we will call **asymptotic stability**. The next pictures show experiments with some **orbits** $A^n \vec{v}$ with different matrices.



ASYMPTOTIC STABILITY. The origin $\vec{0}$ is invariant under a linear map $T(\vec{x}) = A\vec{x}$. It is called **asymptotically stable** if $A^n(\vec{x}) \rightarrow \vec{0}$ for all $\vec{x} \in \mathbb{R}^n$.

EXAMPLE. Let $A = \begin{bmatrix} p & -q \\ q & p \end{bmatrix}$ be a dilation rotation matrix. Because multiplication with such a matrix is analogue to the multiplication with a complex number $z = p + iq$, the matrix A^n corresponds to a multiplication with $(p + iq)^n$. Since $|(p + iq)^n| = |p + iq|^n$, the origin is asymptotically stable if and only if $|p + iq| < 1$. Because $\det(A) = |p + iq|^2 = |z|^2$, rotation-dilation matrices A have an asymptotically stable origin if and only if $|\det(A)| < 1$. Dilation-rotation matrices $\begin{bmatrix} p & -q \\ q & p \end{bmatrix}$ have eigenvalues $p \pm iq$ and can be diagonalized in the complex.

EXAMPLE. If a matrix A has an eigenvalue $|\lambda| \geq 1$ to an eigenvector \vec{v} , then $A^n \vec{v} = \lambda^n \vec{v}$, whose length is $|\lambda|^n$ times the length of \vec{v} . So, we have no asymptotic stability if an eigenvalue satisfies $|\lambda| \geq 1$.

STABILITY. The book also writes "stable" for "asymptotically stable". This is ok to abbreviate. Note however that the commonly used term "stable" also includes linear maps like rotations, reflections or the identity. It is therefore preferable to leave the attribute "asymptotic" in front of "stable".

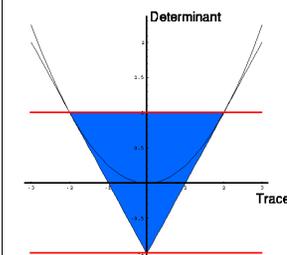
ROTATIONS. Rotations $\begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$ have the eigenvalue $\exp(\pm i\phi) = \cos(\phi) + i\sin(\phi)$ and are not asymptotically stable.

DILATIONS. Dilations $\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$ have the eigenvalue r with algebraic and geometric multiplicity 2. Dilations are asymptotically stable if $|r| < 1$.

CRITERION. A linear dynamical system $x \mapsto Ax$ has an asymptotically stable origin if and only if all its eigenvalues have an absolute value < 1 .

PROOF. We have already seen in Example 3, that if one eigenvalue satisfies $|\lambda| > 1$, then the origin is not asymptotically stable. If $|\lambda_i| < 1$ for all i and all eigenvalues are different, there is an eigenbasis v_1, \dots, v_n . Every x can be written as $x = \sum_{j=1}^n x_j v_j$. Then, $A^n x = A^n (\sum_{j=1}^n x_j v_j) = \sum_{j=1}^n x_j \lambda_j^n v_j$ and because $|\lambda_j|^n \rightarrow 0$, there is stability. The proof of the general (nondiagonalizable) case reduces to the analysis of shear dilations.

THE 2-DIMENSIONAL CASE. The characteristic polynomial of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $f_A(\lambda) = \lambda^2 - \text{tr}(A)\lambda + \det(A)$. If $c \neq 0$, the eigenvalues are $\lambda_{\pm} = \text{tr}(A)/2 \pm \sqrt{(\text{tr}(A)/2)^2 - \det(A)}$. If the **discriminant** $(\text{tr}(A)/2)^2 - \det(A)$ is nonnegative, then the eigenvalues are real. This happens below the parabola, where the discriminant is zero.



CRITERION. In two dimensions we have asymptotic stability if and only if $(\text{tr}(A), \det(A))$ is contained in the **stability triangle** bounded by the lines $\det(A) = 1$, $\det(A) = \text{tr}(A) - 1$ and $\det(A) = -\text{tr}(A) - 1$.

PROOF. Write $T = \text{tr}(A)/2$, $D = \det(A)$. If $|D| \geq 1$, there is no asymptotic stability. If $\lambda = T + \sqrt{T^2 - D} = \pm 1$, then $T^2 - D = (\pm 1 - T)^2$ and $D = 1 \pm 2T$. For $D \leq -1 + |2T|$ we have a real eigenvalue ≥ 1 . The conditions for stability is therefore $D > |2T| - 1$. It implies automatically $D > -1$ so that the triangle can be described shortly as $|\text{tr}(A)| - 1 < \det(A) < 1$.

EXAMPLES.

- 1) The matrix $A = \begin{bmatrix} 1 & 1/2 \\ -1/2 & 1 \end{bmatrix}$ has determinant $5/4$ and trace 2 and the origin is unstable. It is a dilation-rotation matrix which corresponds to the complex number $1 + i/2$ which has an absolute value > 1 .
- 2) A rotation A is never asymptotically stable: $\det(A) = 1$ and $\text{tr}(A) = 2\cos(\phi)$. Rotations are the upper side of the **stability triangle**.
- 3) A dilation is asymptotically stable if and only if the scaling factor has norm < 1 .
- 4) If $\det(A) = 1$ and $\text{tr}(A) < 2$ then the eigenvalues are on the unit circle and there is no asymptotic stability.
- 5) If $\det(A) = -1$ (like for example Fibonacci) there is no asymptotic stability. For $\text{tr}(A) = 0$, we are a corner of the stability triangle and the map is a reflection, which is not asymptotically stable neither.

SOME PROBLEMS.

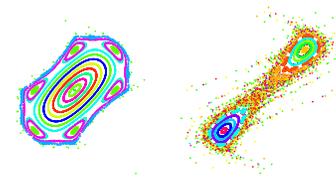
- 1) If A is a matrix with asymptotically stable origin, what is the stability of 0 with respect to A^T ?
- 2) If A is a matrix which has an asymptotically stable origin, what is the stability with respect to A^{-1} ?
- 3) If A is a matrix which has an asymptotically stable origin, what is the stability with respect to A^{100} ?

ON THE STABILITY QUESTION.

For general dynamical systems, the question of stability can be very difficult. We deal here only with linear dynamical systems, where the eigenvalues determine everything. For nonlinear systems, the story is not so simple even for simple maps like the Henon map. The questions go deeper: it is for example not known, whether our solar system is stable. We don't know whether in some future, one of the planets could get expelled from the solar system (this is a mathematical question because the escape time would be larger than the life time of the sun). For other dynamical systems like the atmosphere of the earth or the stock market, we would really like to know what happens in the near future...



A pioneer in stability theory was Aleksandr Lyapunov (1857-1918). For nonlinear systems like $x_{n+1} = gx_n - x_n^3 - x_{n-1}$ the stability of the origin is nontrivial. As with Fibonacci, this can be written as $(x_{n+1}, x_n) = (gx_n - x_n^2 - x_{n-1}, x_n) = A(x_n, x_{n-1})$ called **cubic Henon map** in the plane. To the right are orbits in the cases $g = 1.5$, $g = 2.5$.



The first case is stable (but proving this requires a fancy theory called KAM theory), the second case is unstable (in this case actually the linearization at $\vec{0}$ determines the picture).

JORDAN NORMAL FORM THEOREM.

Every $n \times n$ matrix A is similar to a matrix

$$\begin{bmatrix} [A_1] & 0 & \dots & 0 \\ 0 & [A_2] & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & & [A_k] \end{bmatrix},$$

where A_i are matrices of the form

$$A_i = \begin{bmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{bmatrix}$$

are matrices called **Jordan blocks**. This theorem gives a complete answer, when two matrices are similar: they must have the same Jordan form. The theorem is beyond this course.

EXAMPLES OF MATRICES IN JORDAN FORM:

1) A generalized shear matrix $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$. It is itself a Jordan block.

2) $\begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$. This matrix has three Jordan blocks. The first one is a 2x2 Jordan block, the second a 1x2 Jordan block, the third again a 2x2 Jordan block.

3) Every diagonal matrix like

$$\begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

is in Jordan normal form. It consists of 5 Jordan blocks.

4)

$$\begin{bmatrix} 5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

is in Jordan normal form. It consists of 4 Jordan blocks. Note that the diagonal elements can be complex and that we can have different blocks with the same diagonal elements. The eigenvalue 5 for example has here 3 Jordan blocks of size 1x1, 2x2 and 3x3.

QUESTION: How many different Jordan normal forms do exist for a 5×5 matrix with eigenvalues 3 and 2 of algebraic multiplicity 3 and 2?

ANSWER: Examples 2) and 3) are examples. There are more.

QUESTION: Is $\begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$ in Jordan normal form?

SYMMETRIC MATRICES

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SYMMETRIC MATRICES. A matrix A with real entries is **symmetric**, if $A^T = A$.

EXAMPLES. $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ is symmetric, $A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$ is not symmetric.

EIGENVALUES OF SYMMETRIC MATRICES. Symmetric matrices A have real eigenvalues.

PROOF. The dot product is extended to complex vectors as $(v, w) = \sum_i \bar{v}_i w_i$. For real vectors it satisfies $(v, w) = v \cdot w$ and has the property $(Av, w) = (v, A^T w)$ for real matrices A and $(\lambda v, w) = \bar{\lambda}(v, w)$ as well as $(v, \lambda w) = \lambda(v, w)$. Now $\bar{\lambda}(v, v) = (\lambda v, v) = (Av, v) = (v, A^T v) = (v, Av) = (v, \lambda v) = \lambda(v, v)$ shows that $\bar{\lambda} = \lambda$ because $(v, v) \neq 0$ for $v \neq 0$.

EXAMPLE. $A = \begin{bmatrix} p & -q \\ q & p \end{bmatrix}$ has eigenvalues $p + iq$ which are real if and only if $q = 0$.

EIGENVECTORS OF SYMMETRIC MATRICES.

Symmetric matrices have an orthonormal eigenbasis if the eigenvalues are all different.

PROOF. Assume $Av = \lambda v$ and $Aw = \mu w$. The relation $\lambda(v, w) = (\lambda v, w) = (Av, w) = (v, A^T w) = (v, Aw) = (v, \mu w) = \mu(v, w)$ is only possible if $(v, w) = 0$ if $\lambda \neq \mu$.

WHY ARE SYMMETRIC MATRICES IMPORTANT? In applications, matrices are often symmetric. For example in **geometry** as **generalized dot products** $v \cdot Av$, or in **statistics** as **correlation matrices** $\text{Cov}[X_k, X_l]$ or in quantum mechanics as **observables** or in **neural networks** as **learning maps** $x \mapsto \text{sign}(Wx)$ or in graph theory as **adjacency matrices** etc. etc. Symmetric matrices play the same role as real numbers do among the complex numbers. Their eigenvalues often have physical or geometrical interpretations. One can also calculate with symmetric matrices like with numbers: for example, we can solve $B^2 = A$ for B if A is symmetric matrix and B is square root of A . This is not possible in general: try to find a matrix B such that $B^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \dots$

RECALL. We have seen when an eigenbasis exists, a matrix A is similar to a diagonal matrix $B = S^{-1}AS$, where $S = [v_1, \dots, v_n]$. Similar matrices have the same characteristic polynomial $\det(B - \lambda) = \det(S^{-1}(A - \lambda)S) = \det(A - \lambda)$ and have therefore the same determinant, trace and eigenvalues. Physicists call the set of eigenvalues also **the spectrum**. They say that these matrices are isospectral. The spectrum is what you "see" (etymologically the name originates from the fact that in quantum mechanics the spectrum of radiation can be associated with eigenvalues of matrices.)

SPECTRAL THEOREM. Symmetric matrices A can be diagonalized $B = S^{-1}AS$ with an orthogonal S .

PROOF. If all eigenvalues are different, there is an eigenbasis and diagonalization is possible. The eigenvectors are all orthogonal and $B = S^{-1}AS$ is diagonal containing the eigenvalues. In general, we can change the matrix A to $A + (C - A)t$ where C is a matrix with pairwise different eigenvalues. Then the eigenvalues are different for all except finitely many t . The orthogonal matrices S_t converges for $t \rightarrow 0$ to an orthogonal matrix S and S diagonalizes A .

WAIT A SECOND ... Why could we not perturb a general matrix A_t to have disjoint eigenvalues and A_t could be diagonalized: $S_t^{-1}A_t S_t = B_t$? The problem is that S_t might become singular for $t \rightarrow 0$.

EXAMPLE 1. The matrix $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ has the eigenvalues $a + b, a - b$ and the eigenvectors $v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. They are orthogonal. The orthogonal matrix $S = [v_1 \ v_2]$ diagonalizes A .

EXAMPLE 2. The 3×3 matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ has 2 eigenvalues 0 to the eigenvectors $[1 \ -1 \ 0]$, $[1 \ 0 \ -1]$ and one eigenvalue 3 to the eigenvector $[1 \ 1 \ 1]$. All these vectors can be made orthogonal and a diagonalization is possible even so the eigenvalues have multiplicities.

SQUARE ROOT OF A MATRIX. How do we find a square root of a given symmetric matrix? Because $S^{-1}AS = B$ is diagonal and we know how to take a square root of the diagonal matrix B , we can form $C = S\sqrt{B}S^{-1}$ which satisfies $C^2 = S\sqrt{B}S^{-1}S\sqrt{B}S^{-1} = SBS^{-1} = A$.

RAYLEIGH FORMULA. We write also $(\vec{v}, \vec{w}) = \vec{v} \cdot \vec{w}$. If $\vec{v}(t)$ is an eigenvector of length 1 to the eigenvalue $\lambda(t)$ of a symmetric matrix $A(t)$ which depends on t , differentiation of $(A(t) - \lambda(t))\vec{v}(t) = 0$ with respect to t gives $(A' - \lambda')v + (A - \lambda)v' = 0$. The symmetry of $A - \lambda$ implies $0 = (v, (A' - \lambda')v) + (v, (A - \lambda)v') = (v, (A' - \lambda')v)$. We see that the **Rayleigh quotient** $\lambda' = (A'v, v)$ is a polynomial in t if $A(t)$ only involves terms t, t^2, \dots, t^m . The formula shows how $\lambda(t)$ changes, when t varies. For example, $A(t) = \begin{bmatrix} 1 & t^2 \\ t^2 & 1 \end{bmatrix}$ has for $t = 2$ the eigenvector $\vec{v} = [1, 1]/\sqrt{2}$ to the eigenvalue $\lambda = 5$. The formula tells that $\lambda'(2) = (A'(2)\vec{v}, \vec{v}) = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix} \vec{v}, \vec{v} = 4$. Indeed, $\lambda(t) = 1 + t^2$ has at $t = 2$ the derivative $2t = 4$.

EXHIBIT. "Where do symmetric matrices occur?" Some informal pointers:

I) PHYSICS:

In **quantum mechanics** a system is described with a vector $v(t)$ which depends on time t . The evolution is given by the **Schrodinger equation** $\dot{v} = i\hbar Lv$, where L is a symmetric matrix and \hbar is a small number called the Planck constant. As for any linear differential equation, one has $v(t) = e^{i\hbar Lt}v(0)$. If $v(0)$ is an eigenvector to the eigenvalue λ , then $v(t) = e^{i\hbar \lambda t}v(0)$. Physical observables are given by symmetric matrices too. The matrix L represents the energy. Given $v(t)$, the value of the observable $A(t)$ is $v(t) \cdot Av(t)$. For example, if v is an eigenvector to an eigenvalue λ of the energy matrix L , then the energy of $v(t)$ is λ .



This is called the **Heisenberg picture**. In order that $v \cdot A(t)v = v(t) \cdot Av(t) = S(t)v \cdot AS(t)v$ we have $A(t) = S(T)^*AS(t)$, where $S^* = \overline{S^T}$ is the correct generalization of the adjoint to complex matrices. $S(t)$ satisfies $S(t)^*S(t) = 1$ which is called **unitary** and the complex analogue of orthogonal. The matrix $A(t) = S(t)^*AS(t)$ has the same eigenvalues as A and is **similar** to A .

II) CHEMISTRY.

The **adjacency matrix** A of a graph with n vertices determines the graph: one has $A_{ij} = 1$ if the two vertices i, j are connected and zero otherwise. The matrix A is symmetric. The eigenvalues λ_j are real and can be used to analyze the graph. One interesting question is to what extent the eigenvalues determine the graph. In chemistry, one is interested in such problems because it allows to make rough computations of the electron density distribution of molecules. In this so called **Hückel theory**, the molecule is represented as a graph. The eigenvalues λ_j of that graph approximate the energies an electron on the molecule. The eigenvectors describe the electron density distribution.



The **Freon molecule** CCl_2F_2 for example has 5 atoms. The adjacency matrix is

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix A has the eigenvalue 0 with multiplicity 3 ($\ker(A)$ is obtained immediately from the fact that 4 rows are the same) and the eigenvalues 2, -2. The eigenvector to the eigenvalue ± 2 is $[\pm 2 \ 1 \ 1 \ 1 \ 1]^T$.

III) STATISTICS.

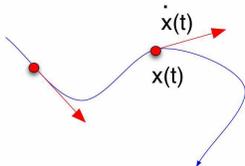
If we have a random vector $X = [X_1, \dots, X_n]$ and $E[X_k]$ denotes the expected value of X_k , then $[A]_{ki} = E[(X_k - E[X_k])(X_i - E[X_i])] = E[X_k X_i] - E[X_k]E[X_i]$ is called the **covariance matrix** of the random vector X . It is a symmetric $n \times n$ matrix. Diagonalizing this matrix $B = S^{-1}AS$ produces new random variables which are **uncorrelated**.

For example, if X is the sum of two dice and Y is the value of the second dice then $E[X] = [(1+1) + (1+2) + \dots + (6+6)]/36 = 7$, you throw in average a sum of 7 and $E[Y] = (1+2 + \dots + 6)/6 = 7/2$. The matrix entry $A_{11} = E[X^2] - E[X]^2 = [(1+1) + (1+2) + \dots + (6+6)]/36 - 7^2 = 35/6$ known as the **variance** of X , and $A_{22} = E[Y^2] - E[Y]^2 = (1^2 + 2^2 + \dots + 6^2)/6 - (7/2)^2 = 35/12$ known as the **variance** of Y and $A_{12} = E[XY] - E[X]E[Y] = 35/12$. The covariance matrix is the symmetric matrix $A = \begin{bmatrix} 35/6 & 35/12 \\ 35/12 & 35/12 \end{bmatrix}$.

CONTINUOUS DYNAMICAL SYSTEMS I

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CONTINUOUS DYNAMICAL SYSTEMS. A differential equation $\frac{d}{dt}\vec{x} = f(\vec{x})$ defines a dynamical system. The solutions is a curve $\vec{x}(t)$ which has the **velocity vector** $f(\vec{x}(t))$ for all t . One often writes \dot{x} instead of $\frac{d}{dt}x$. So, we know a formula for the tangent at each point. The aim is to find a curve $\vec{x}(t)$ which starts at a given point $\vec{v} = \vec{x}(0)$ and has the prescribed directions at each time.



IN ONE DIMENSION. A system $\dot{x} = g(x, t)$ is the general differential equation in one dimensions. Examples:

- If $\dot{x} = g(t)$, then $x(t) = \int_0^t g(t) dt$. Example: $\dot{x} = \sin(t), x(0) = 0$ has the solution $x(t) = \cos(t) - 1$.
- If $\dot{x} = h(x)$, then $dx/h(x) = dt$ and so $t = \int_0^x dx/h(x) = H(x)$ so that $x(t) = H^{-1}(t)$. Example: $\dot{x} = \frac{1}{\cos(x)}$ with $x(0) = 0$ gives $dx \cos(x) = dt$ and after integration $\sin(x) = t + C$ so that $x(t) = \arcsin(t + C)$. From $x(0) = 0$ we get $C = \pi/2$.
- If $\dot{x} = g(t)/h(x)$, then $H(x) = \int_0^x h(x) dx = \int_0^t g(t) dt = G(t)$ so that $x(t) = H^{-1}(G(t))$. Example: $\dot{x} = \sin(t)/x^2, x(0) = 0$ gives $dx x^2 = \sin(t) dt$ and after integration $x^3/3 = -\cos(t) + C$ so that $x(t) = (3C - 3\cos(t))^{1/3}$. From $x(0) = 0$ we obtain $C = 1$.

Remarks:

- 1) In general, we have no closed form solutions in terms of known functions. The solution $x(t) = \int_0^t e^{-t^2} dt$ of $\dot{x} = e^{-t^2}$ for example can not be expressed in terms of functions exp, sin, log, $\sqrt{\cdot}$ etc but it can be solved using Taylor series: because $e^{-t^2} = 1 - t^2 + t^4/2! - t^6/3! + \dots$ taking coefficient wise the anti-derivatives gives: $x(t) = t - t^3/3 + t^5/(32) - t^7/(73!) + \dots$
- 2) The system $\dot{x} = g(x, t)$ can be written in the form $\dot{\vec{x}} = f(\vec{x})$ with $\vec{x} = (x, t)$. $\frac{d}{dt} \begin{bmatrix} x \\ t \end{bmatrix} = \begin{bmatrix} g(x, t) \\ 1 \end{bmatrix}$.

ONE DIMENSIONAL LINEAR DIFFERENTIAL EQUATIONS. The most general linear system in one dimension is $\dot{x} = \lambda x$. It has the solution $x(t) = e^{\lambda t} x(0)$. This differential equation appears

- as **population models** with $\lambda > 0$. The birth rate of the population is proportional to its size.
- as **radioactive decay** with $\lambda < 0$. The decay rate is proportional to the number of atoms.

LINEAR DIFFERENTIAL EQUATIONS IN A NUTSHELL: Linear dynamical systems have the form $\dot{x} = Ax$, where A is a matrix. The origin $\vec{0}$ is always an **equilibrium point**: if $\vec{x}(0) = \vec{0}$, then $\vec{x}(t) = \vec{0}$ for all t . In general, we look for a solution $\vec{x}(t)$ for a given initial point $\vec{x}(0) = \vec{v}$. Here are three different ways to get a closed form solution:

- If $B = S^{-1}AS$ is diagonal with the eigenvalues $\lambda_j = a_j + ib_j$, then $y = S^{-1}x$ satisfies $y(t) = e^{Bt}$ and therefore $y_j(t) = e^{\lambda_j t} y_j(0) = e^{a_j t} e^{ib_j t} y_j(0)$. The solutions in the original coordinates are $x(t) = Sy(t)$.
- If \vec{v}_i are the **eigenvectors** to the eigenvalues λ_i , and $\vec{v} = c_1 \vec{v}_1 + \dots + c_n \vec{v}_n$, then $\vec{x}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + \dots + c_n e^{\lambda_n t} \vec{v}_n$ is a closed formula for the solution of $\frac{d}{dt}\vec{x} = A\vec{x}, \vec{x}(0) = \vec{v}$.
- Linear differential equations can also be solved as in one dimensions: the general solution of $\dot{x} = Ax, \vec{x}(0) = \vec{v}$ is $x(t) = e^{At}\vec{v} = (1 + At + A^2t^2/2! + \dots)\vec{v}$, because $\dot{x}(t) = A + 2A^2t/2! + \dots = A(1 + At + A^2t^2/2! + \dots)\vec{v} = Ae^{At}\vec{v} = Ax(t)$. This solution does not provide us with much insight however.

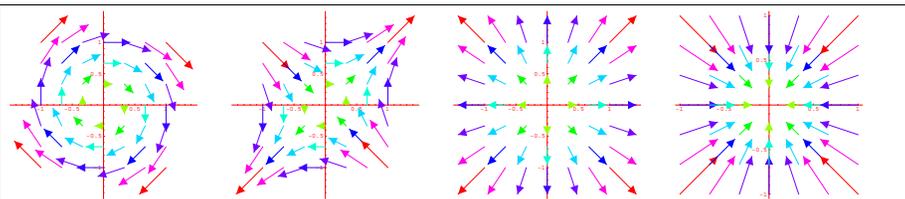
EXAMPLE. Find a closed formula for the solution of the system

$$\begin{aligned} \dot{x}_1 &= x_1 + 2x_2 \\ \dot{x}_2 &= 4x_1 + 3x_2 \end{aligned}$$

with $\vec{x}(0) = \vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The system can be written as $\dot{x} = Ax$ with $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$. The matrix A has the eigenvector $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ to the eigenvalue -1 and the eigenvector $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ to the eigenvalue 5 .

Because $A\vec{v}_1 = -\vec{v}_1$, we have $\vec{v}_1(t) = e^{-t}\vec{v}$. Because $A\vec{v}_2 = 5\vec{v}_2$, we have $\vec{v}_2(t) = e^{5t}\vec{v}$. The vector \vec{v} can be written as a linear-combination of \vec{v}_1 and \vec{v}_2 : $\vec{v} = \frac{1}{3}\vec{v}_2 + \frac{2}{3}\vec{v}_1$. Therefore, $\vec{x}(t) = \frac{1}{3}e^{5t}\vec{v}_2 + \frac{2}{3}e^{-t}\vec{v}_1$.

PHASE PORTRAITS. For differential equations $\dot{x} = f(x)$ in two dimensions, one can **draw the vector field** $x \mapsto f(x)$. The solution curve $x(t)$ is tangent to the vector $f(x(t))$ everywhere. The phase portraits together with some solution curves reveal much about the system. Examples are



UNDERSTANDING A DIFFERENTIAL EQUATION. The closed form solution like $x(t) = e^{At}x(0)$ for $\dot{x} = Ax$ does not give us much insight what happens. One wants to understand the solution quantitatively. We want to understand questions like: What happens in the long term? Is the origin stable? Are there periodic solutions. Can one decompose the system into simpler subsystems? We will see that **diagonalisation** allows to **understand the system**. By decomposing it into one-dimensional linear systems, it can be analyzed separately. In general "understanding" can mean different things:

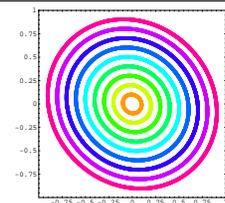
- | | |
|--|--|
| <ul style="list-style-type: none"> Plotting phase portraits. Computing solutions numerically and estimate the error. Finding special solutions. Predicting the shape of some orbits. Finding regions which are invariant. | <ul style="list-style-type: none"> Finding special closed form solutions $x(t)$. Finding a power series $x(t) = \sum_n a_n t^n$ in t. Finding quantities which are unchanged along the flow (called "Integrals"). Finding quantities which increase along the flow (called "Lyapunov functions"). |
|--|--|

LINEAR STABILITY. A linear dynamical system $\dot{x} = Ax$ with diagonalizable A is linearly stable if and only if $a_j = \text{Re}(\lambda_j) < 0$ for all eigenvalues λ_j of A .

PROOF. We see that from the explicit solutions $y_j(t) = e^{a_j t} e^{ib_j t} y_j(0)$ in the basis consisting of eigenvectors. Now, $y(t) \rightarrow 0$ if and only if $a_j < 0$ for all j and $x(t) = Sy(t) \rightarrow 0$ if and only if $y(t) \rightarrow 0$.

RELATION WITH DISCRETE TIME SYSTEMS. From $\dot{x} = Ax$, we obtain $x(t+1) = Bx(t)$, with the matrix $B = e^A$. The eigenvalues of B are $\mu_j = e^{\lambda_j}$. Now $|\mu_j| < 1$ if and only if $\text{Re}(\lambda_j) < 0$. The criterium for linear stability of discrete dynamical systems is compatible with the criterium for linear stability of $\dot{x} = Ax$.

EXAMPLE 1. The system $\dot{x} = y, \dot{y} = -x$ can in vector form $v = (x, y)$ be written as $\dot{v} = Av$, with $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The matrix A has the eigenvalues $i, -i$. After a coordinate transformation $w = S^{-1}v$ we get with $w = (a, b)$ the differential equations $\dot{a} = ia, \dot{b} = -ib$ which has the solutions $a(t) = e^{it}a(0), b(t) = e^{-it}b(0)$. The original coordinates satisfy $x(t) = \cos(t)x(0) - \sin(t)y(0), y(t) = \sin(t)x(0) + \cos(t)y(0)$.



CONTINUOUS DYNAMICAL SYSTEMS II

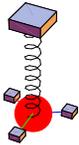
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COMPLEX LINEAR 1D CASE. $\dot{x} = \lambda x$ for $\lambda = a + ib$ has solution $x(t) = e^{at}e^{ibt}x(0)$ and length $|x(t)| = e^{at}|x(0)|$. Application: the differential equation $\dot{z} = iz$ has the solution e^{it} as well as the $\cos(t) + i \sin(t)$. Because the solutions of the differential equation are unique with $z(0) = 1$, we have an other verification of the **Euler formula** $e^{it} = \cos(t) + i \sin(t)$.

THE HARMONIC OSCILLATOR: $\ddot{x} = -cx$ is solved by $x(t) = \cos(\sqrt{ct})x(0) + \sin(\sqrt{ct})\dot{x}(0)/\sqrt{c}$.
DERIVATION. $\dot{x} = y, \dot{y} = -\lambda x$ and in matrix form as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ \lambda & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$$

and because A has eigenvalues $\pm i\sqrt{\lambda}$, the new coordinates move as $a(t) = e^{i\sqrt{ct}}a(0)$ and $b(t) = e^{-i\sqrt{ct}}b(0)$. Writing this in the original coordinates $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} a(t) \\ b(t) \end{bmatrix}$ and fixing the constants gives $x(t), y(t)$.



EXAMPLE. THE SPINNER. The spinner is a rigid body attached to a spring aligned around the z-axis. The body can rotate around the z-axis and bounce up and down. The two motions are coupled in the following way: when the spinner winds up in the same direction as the spring, the spring gets tightened and the body gets a lift. If the spinner winds up to the other direction, the spring becomes more relaxed and the body is lowered. Instead of reducing the system to a 4D first order system, system $\frac{d^2}{dt^2}\vec{x} = A\vec{x}$, we will keep the second time derivative and diagonalize the 2D system $\frac{d^2}{dt^2}\vec{x} = A\vec{x}$, where we know how to solve the one dimensional case $\frac{d^2}{dt^2}v = -\lambda v$ as $v(t) = A \cos(\sqrt{\lambda}t) + B \sin(\sqrt{\lambda}t)$ with constants A, B depending on the initial conditions, $v(0), \dot{v}(0)$.

THE DIFFERENTIAL EQUATIONS OF THE SPINNER.

x is the angle and y the height of the body. We put the coordinate system so that $y = 0$ is the point, where the body stays at rest if $x = 0$. We assume that if the spring is wound up with an angle x , this produces an upwards force x and a momentum force $-3x$. We furthermore assume that if the body is at position y , then this produces a momentum y onto the body and an upwards force y . The differential equations

$$\begin{aligned} \ddot{x} &= -3x + y & \text{can be written as } \ddot{v} &= Av = \begin{bmatrix} -3 & 1 \\ 1 & -1 \end{bmatrix} v. \\ \ddot{y} &= -y + x \end{aligned}$$

FINDING GOOD COORDINATES $w = S^{-1}v$ is obtained with getting the eigenvalues and eigenvectors of A : $\lambda_1 = -2 - \sqrt{2}, \lambda_2 = -2 + \sqrt{2}$ $v_1 = \begin{bmatrix} -1 - \sqrt{2} \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 + \sqrt{2} \\ 1 \end{bmatrix}$ so that $S = \begin{bmatrix} -1 - \sqrt{2} & -1 + \sqrt{2} \\ 1 & 1 \end{bmatrix}$.

SOLVE THE SYSTEM $\ddot{a} = \lambda_1 a, \ddot{b} = \lambda_2 b$ IN THE GOOD COORDINATES $\begin{bmatrix} a \\ b \end{bmatrix} = S^{-1} \begin{bmatrix} x \\ y \end{bmatrix}$.
 $a(t) = A \cos(\omega_1 t) + B \sin(\omega_1 t), \omega_1 = \sqrt{-\lambda_1}, b(t) = C \cos(\omega_2 t) + D \sin(\omega_2 t), \omega_2 = \sqrt{-\lambda_2}$.

THE SOLUTION IN THE ORIGINAL COORDINATES. $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} a(t) \\ b(t) \end{bmatrix}$. At $t = 0$ we know $x(0), y(0), \dot{x}(0), \dot{y}(0)$. This fixes the constants in $x(t) = A_1 \cos(\omega_1 t) + B_1 \sin(\omega_1 t) + A_2 \cos(\omega_2 t) + B_2 \sin(\omega_2 t)$. The curve $(x(t), y(t))$ traces a Lyssajoux curve:

ASYMPTOTIC STABILITY $\dot{x} = Ax$ is asymptotically stable if and only if $\text{Re}(\lambda_i) < 0$ for all i .

ASYMPTOTIC STABILITY IN 2D A linear system $\dot{x} = Ax$ in the 2D plane is asymptotically stable if and only if $\det(A) > 0$ and $\text{tr}(A) < 0$.

PROOF. If both eigenvalues λ_1, λ_2 are real, then both being negative is equivalent to $\lambda_1 \lambda_2 = \det(A) > 0$ and $\text{tr}(A) = \lambda_1 + \lambda_2 < 0$. If $\lambda_1 = a + ib, \lambda_2 = a - ib$, then a negative a is equivalent to $\lambda_1 + \lambda_2 = 2a < 0$ and $\lambda_1 \lambda_2 = a^2 + b^2 > 0$.

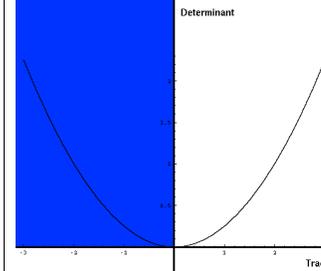
ASYMPTOTIC STABILITY COMPARISON OF DISCRETE AND CONTINUOUS SITUATION.

The trace and the determinant are independent of the basis, they can be computed fast, and are real if A is real. It is therefore convenient to determine the region in the $\text{tr} - \det$ -plane, where continuous or discrete dynamical systems are asymptotically stable. While the continuous dynamical system is related to a discrete system, it is important not to mix these two situations up.

Continuous dynamical system.

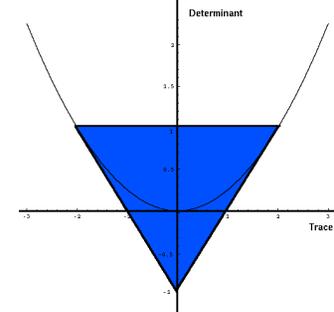
Discrete dynamical system.

Stability of $\dot{x} = Ax$ ($x(t+1) = e^A x(t)$).



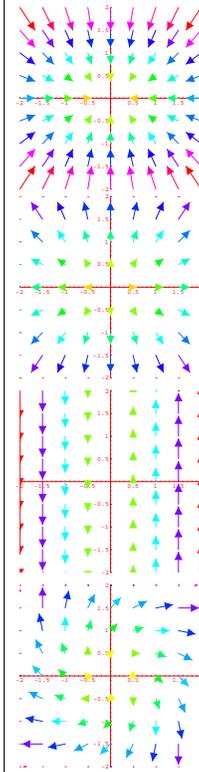
Stability in $\det(A) > 0, \text{tr}(A) < 0$
 Stability if $\text{Re}(\lambda_1) < 0, \text{Re}(\lambda_2) < 0$.

Stability of $x(t+1) = Ax$



Stability in $|\text{tr}(A)| < 1 < \det(A) < 1$
 Stability if $|\lambda_1| < 1, |\lambda_2| < 1$.

PHASE-PORTRAITS. (In two dimensions we can plot the vector field, draw some trajectories)

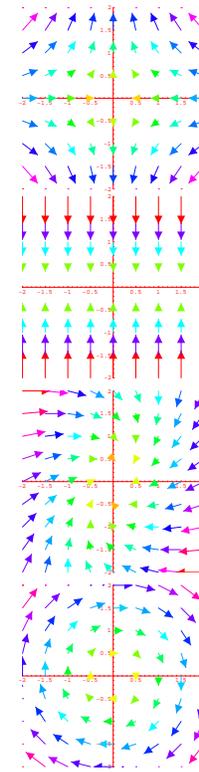


$\lambda_1 < 0$
 $\lambda_2 < 0$,
 i.e. $A = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$

$\lambda_1 > 0$
 $\lambda_2 > 0$,
 i.e. $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

$\lambda_1 = 0$
 $\lambda_2 = 0$,
 i.e. $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$\lambda_1 = a + ib, a > 0$
 $\lambda_2 = a - ib$,
 i.e. $A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$



$\lambda_1 < 0$
 $\lambda_2 > 0$,
 i.e. $A = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$

$\lambda_1 = 0$
 $\lambda_2 < 0$,
 i.e. $A = \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix}$

$\lambda_1 = a + ib, a < 0$
 $\lambda_2 = a - ib$,
 i.e. $A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$

$\lambda_1 = ib$
 $\lambda_2 = -ib$,
 i.e. $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

DIFFERENTIAL EQUATIONS I 21b, O. Knill

Lets look at a differential equation

$$\frac{d}{dt}\vec{x} = A\vec{x}$$

with

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} .$$

and $\vec{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. If A were diagonalizable, we could get closed form solutions

$$\vec{x}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2 .$$

a) In our specific case, the differential equation can be written out as

$$\begin{aligned} \dot{x} &= \dots \\ \dot{y} &= \dots \end{aligned}$$

b) Can you reduce this to a differential equation of one variables?

$$\dot{x} = x + \dots$$

c) Can you guess a function $x(t)$ which satisfies this equation?

$$x(t) = \dots$$

We will learn later how to deal with differential equations of more general form later and in a systematic way.

REVIEW ROTATION DILATION MATRICES.

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

has the eigenvalue $\lambda_+ = a + ib$ to the eigenvector $v_+ = \begin{bmatrix} i \\ 1 \end{bmatrix}$ and the eigenvalue $\lambda_- = a - ib$ to the eigenvector $v_- = \begin{bmatrix} -i \\ 1 \end{bmatrix}$.

For the FIRST ORDER LINEAR DIFFERENTIAL EQUATION

$$\begin{aligned} \frac{d}{dt}x &= ax - by \\ \frac{d}{dt}y &= bx + ay \end{aligned}$$

we have $\frac{d}{dt}\vec{v}_- = (a + ib)\vec{v}_-$ and so

$$\vec{v}_+(t) = e^{at}e^{+ibt}\vec{v}_+$$

Similar, we get

$$\vec{v}_-(t) = e^{at}e^{-ibt}\vec{v}_-$$

With a general initial condition $\vec{x}(0) = c_+\vec{v}_+ + c_-\vec{v}_-$, we have then $\vec{x}(t) = c_+\vec{v}_+(t) + c_-\vec{v}_-(t) = e^{at}(c_+e^{+ibt} + c_-e^{-ibt})$. For $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we have $c_+ = 1/2, c_- = 1/2$ we get with that initial condition $\vec{x}(t) = e^{at}(\vec{v}_+(t) + \vec{v}_-(t))/2 = e^{at} \begin{bmatrix} \cos(bt) \\ \sin(bt) \end{bmatrix}$. For $\vec{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ we have $c_+ = 1/(2i), c_- = -1/(2i)$ so that $\vec{x}(t) = e^{at}(\vec{v}_+(t) - \vec{v}_-(t))/(2i) = e^{at} \begin{bmatrix} -\sin(bt) \\ \cos(bt) \end{bmatrix}$.

The initial value problem

$$\begin{aligned} \frac{d}{dt}x &= ax - by \\ \frac{d}{dt}y &= bx + ay \end{aligned}$$

with $x(0) = A, y(0) = B$ has the explicit solutions

$$\vec{x}(t) = e^{at} \begin{bmatrix} A \cos(bt) - B \sin(bt) \\ B \cos(bt) + A \sin(bt) \end{bmatrix}$$

HOW DOES THE PHASE PORTRAIT LOOK LIKE IF $a = 0$?

HOW DOES THE PHASE PORTRAIT LOOK LIKE IF $a < 0$ and $b = 0$?

HOW DOES THE PHASE PORTRAIT LOOK LIKE IF $a > 0$ and $b = 0$?

SKETCH THE PHASE PORTRAIT IN THE CASE $a < 0$ and $b > 0$.

SKETCH THE PHASE PORTRAIT IN THE CASE $a < 0$ and $b < 0$.

SKETCH THE PHASE PORTRAIT IN THE CASE $a > 0$ and $b > 0$.

SKETCH THE PHASE PORTRAIT IN THE CASE $a > 0$ and $b < 0$.

A single pendulum of small amplitude is described by the **harmonic oscillator** differential equation:

$$\ddot{x} = -\omega^2 x .$$

The acceleration = force pushes in the opposite direction. The solution of this important differential equation is

$$x(t) = x(0) \cos(\omega t) + x'(0) \sin(\omega t) .$$

The frequency c depends on the length l of the pendulum and the gravitational constant g . For the pendulum it is $\omega = \sqrt{g/l}$.

Assume $\omega = 1$ for now and assume two penduli are coupled. We get a system of differential equations

$$\begin{aligned} \ddot{x} &= -x + c(y - x) \\ \ddot{y} &= -y - c(y - x) \end{aligned}$$

Find the matrix such that $\dot{v} = Av$ with $v = \begin{bmatrix} x \\ y \end{bmatrix}$. Find the eigenvalues of A and find $v(t)$.