4.2.30 Linear, since $T(f(t)+g(t)) = t(f'(t)+g'(t)) = t(f'(t)) + t(g'(t))$ equals

$T(f(t)) + T(g(t)) = t(f'(t)) + t(g'(t))$, and $T(kf(t)) = t(kf'(t)) = kt(f'(t))$

equals $kT(f(t)) = kt(f'(t))$.

No, T isn't an isomorphism, since the constant function $f(t) = 1$ is in $\ker(T)$.

4.2.56 Note that $T(a+bt+ct^2) = bt + 2ct^2$. Thus the kernel consists of all constant polynomials $f(t) = a$ (when $b = c = 0$), and the nullity is 1. The image consists of all polynomials of the form $f(t) = pt + qt^2$, and the rank is 2.

4.2.60 Note that $T(a+bt+ct^2) = \begin{bmatrix} a+7b+49c \\ a+11b+121c \end{bmatrix}$. To find the kernel, solve the linear system $\begin{bmatrix} a+7b+49c \\ a+11b+121c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The solution is $a = 77c,\ b = -18c$, so that the kernel consists of all polynomials of the form $f(t) = c(77 - 18t + t^2) = c(t-11)(t-7)$. You can also see directly that the quadratic polynomials $f(t)$ with $f(7) = f(11) = 0$ are of this form. The nullity is 1. The image consists of all of $\mathbb{R}^2$, so that the rank is 2.

4.2.66 The kernel of T consists of all smooth functions $f(t)$ such that

$T(f(t)) = f(t) - f'(t) = 0$, or $f'(t) = f(t)$. As you may recall from a

discussion of exponential functions in calculus, those are the functions of the

form $f(t) = Ce^t$, where C is a constant. Thus the nullity of T is 1.

9.3.6 Using Theorem 9.3.13, $f(t) = e^{2t}\int e^{-2t}e^{2t}\,dt = e^{2t}\int dt = e^{2t}(t+C)$, where C is an arbitrary constant.

4.2.78 a Check all the conditions in Definition 4.1.1. A basis is 2.

b $T(x \oplus y) = T(xy) = \ln(xy) = \ln(x) + \ln(y) = T(x) + T(y)$ and

$T(k \odot x) = T(x^k) = \ln(x^k) = k\ln(x) = kT(x)$.

The inverse of T is $L(y) = e^y$, so that T is indeed an isomorphism.

Ch 4.TF.44 F; We can construct as many linearly independent elements in $\ker(T)$ as we want, for example, the polynomials $f(t) = t^n - \frac{1}{n+1}$, for all positive integers n.

Ch 4.TF.58 F; If A is a scalar multiple of I_2, then all 2×2 matrices commute with A, so that the space of commuting matrices is 4 - dimensional. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ fails to be a scalar multiple of I_2, consider the equation $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x & y \\ z & t \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, which amounts to the system $cy - bz = 0, bx + (d-a)y - bt = 0, cx + (d-a)z - ct = 0$. If $b \neq 0$, then the first two equations are independent; if $c \neq 0$, then the first and the third equation are independent; and if $a \neq d$, then the second and the third equation are independent. Thus the rank of the system is at least two and the solution space is at most two-dimensional. (The solution space is in fact two -dimensional, since A and I_2 are independent solutions.)