Final Exam Practice I, May 7, 2015 **Math 21b, Spring 2015**

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- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) (20 points) True or False? No justifications are needed.

1)	T	F	The matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ is invertible.
2)	T	F	The difference of two eigenvectors v_1, v_2 of a matrix A is always an eigenvector of A.
3)	T	F	For any invertible 7×7 matrix A, the column space of A is an eigenspace of A.
4)	T	F	A 2×2 matrix A is diagonalizable then the matrix A^2 is diagonalizable.
5)	T	F	The function $f(x) = x\sin(x)$ has a Fourier expansion which is a cos series.
6)	T	F	The space of smooth functions satisfying the equation $f(x) = f(1 + f(x)))$ form a linear space.
7)	T	F	The sum of all geometric multiplies of a 5×5 matrix is smaller or equal than the sum of all algebraic multiplicities.
8)	T	F	A sum of a 2×2 rotation matrix and a 2×2 reflection matrix is an orthogonal matrix.
9)	T	F	A discrete dynamical system $x(t+1) = Ax(t)$ with a 4×4 matrix for which all eigenvalues of A are negative, is asymptotically stable.
10)	T	F	There is a matrix A such that $x'(t) = Ax(t)$ and $x(t+1) = Ax(t)$ are both asymptotically stable dynamical systems.
11)	T	F	An orthogonal rotation in the plane composed with an orthogonal projection in the plane is always an orthogonal projection in the plane.
12)	T	F	The function $f(x,t) = e^{-t}\sin(t) - e^{-25t}\sin(5t)$ satisfies the heat equation $f_t = f_{xx}$
13)	T	F	For a 20×10 matrix A, the equation $Ax = b$ has either zero or infinitely many solutions.
14)	T	F	The Jacobian matrix at a equilibrium point $(0,0)$ of a nonlinear system $x' = x + y^2$, $y' = x - y^2$ is invertible.
15)	T	F	There are matrices which are diagonalizable over the complex numbers but not over the reals.
16)	T	F	$\|\|\sin(5x) - \sin(10x)\|\| = 2$, where the length $\|\|f\|\|$ of f is defined by the inner product $\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx$.
17)	T	F	The sum of two eigenvalues of a non-invertible 3×3 matrix A is an eigenvalue of A.
18)	T	F	For the differential equation $x''(t)+x(t) = \sin(t)$, all solutions are bounded.
19)	T	F	If a 2×2 matrix A is similar to a 2×2 matrix B, then the transposed matrix A^T is similar to B^T.
20)	T	F	A 3×3 matrix A is invertible if and only if all its eigenvalues are positive.

Problem 2) (10 points) no justifications needed

a) (5 points) Draw with your pen a connection between any pair of matrices which have the same eigenvalues and which are also similar.

$$B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$$

b) (5 points) Match the initial value problem with their solutions. Each function matches exactly one of the differential equations. The initial condition is $f(0) = 1$ $f'(0) = 0$ in all cases.

enter A-D	initial value problem
	$f''(t) + f(t) = 2\sin(t)$
	$f''(t) + f(t) = 6\sin(2t)$
	$f''(t) - f(t) = 2\sin(t)$
	$f''(t) - f(t) = 6\sin(2t)$

label	solution
A)	$f(t) = \cos(t)_+ \sin(t) - t\cos(t)$
B)	$f(t) = -0.7e^{-t} + 1.7e^{t} - 1.2\sin(2t)$
C)	$f(t) = e^{t} - \sin(t)$
D)	$f(t) = \cos(t) + 4\sin(t) - 2\sin(2t)$

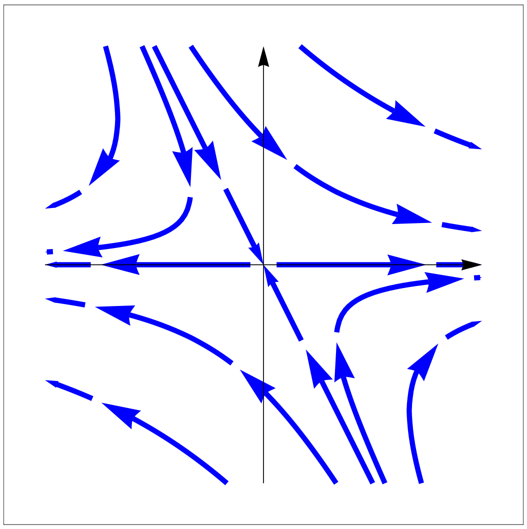
Problem 3) (10 points) no justifications needed

a) (5 points) Match the discrete dynamical systems $x(t+1) = Ax(t)$ with the phase portraits. Enter a)-d).

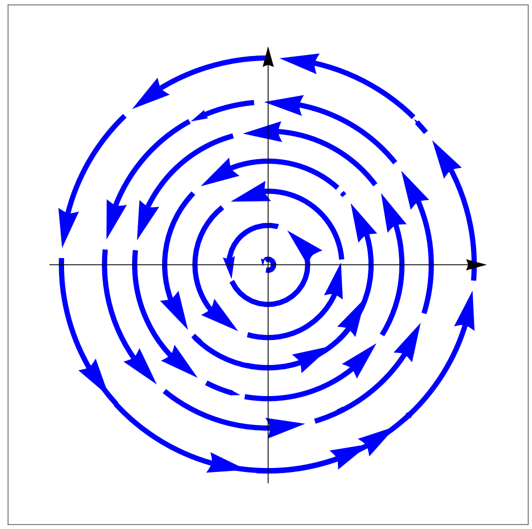
matrix	$x(t+1) = Ax$
$A = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$	
$A = \begin{bmatrix} \cos(1) & -\sin(1) \\ \sin(1) & \cos(1) \end{bmatrix}$	
$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$	
$A = \begin{bmatrix} 2 & -1 \\ 0 & 1/2 \end{bmatrix}$	

b) (5 points) Match the continuous dynamical systems $x' = Ax(t)$ with the phase portraits. Enter a)-d).

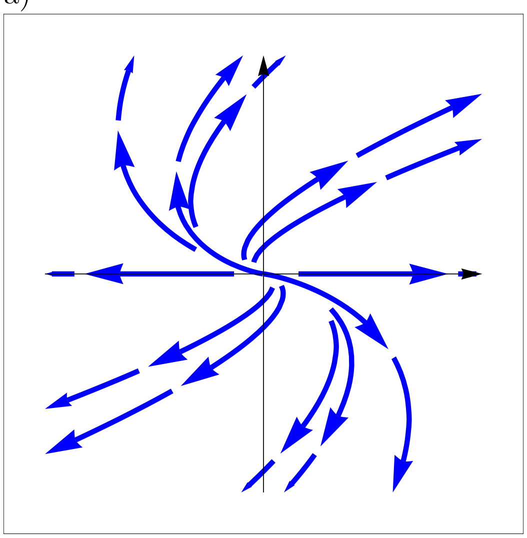
matrix	$\frac{d}{dt}x(t) = Ax(t)$
$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	
$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$	
$A = \begin{bmatrix} 2 & 2 \\ 0 & -2 \end{bmatrix}$	
$A = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$	



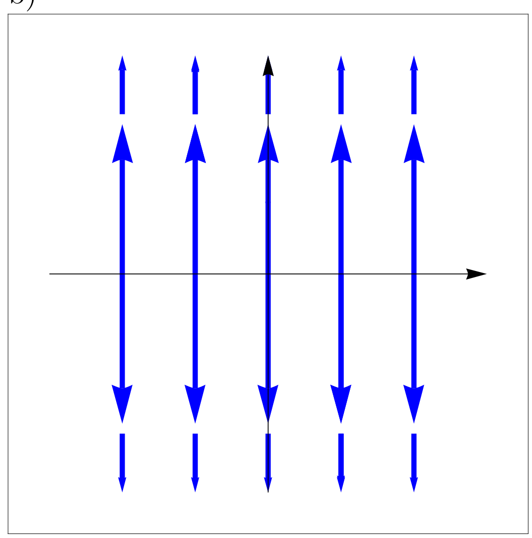
a)



b)



c)



d)

Problem 4) (10 points)

We aim to find all the solutions of the following system of linear equations.

$$\left| \begin{array}{ccccccccccccc} x & + & 2y & + & 3z & + & 3u & + & 2v & + & w & = 3 \\ & & y & + & 2z & + & 2u & + & v & & & = 2 \\ & & & & z & + & u & & & & & = 1 \end{array} \right|$$

a) (2 points) Rewrite the system in matrix form $A\vec{x} = \vec{b}$.

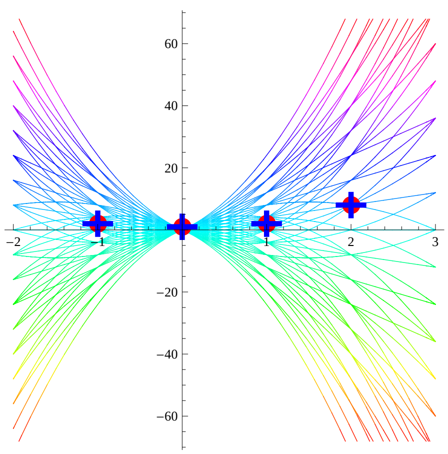
b) (5 points) Row reduce the augmented matrix $[A|b]$.

c) (3 points) Write down the general solution.

Problem 5) (10 points)

Using the least square method, find the best parabolic function of the form $ax + bx^2 = y$ which fits the data points

$$(1, 2), (0, 1), (2, 8), (-1, 1) .$$



Problem 6) (10 points)

a) (5 points) Find all the eigenvalues and eigenvectors of the matrix

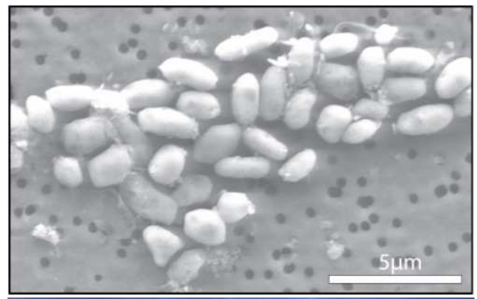
$$A = \begin{bmatrix} 3 & 9 & 9 & 9 \\ 9 & 3 & 9 & 9 \\ 9 & 9 & 3 & 9 \\ 9 & 9 & 9 & 3 \end{bmatrix} .$$

b) (5 points) Find all the eigenvalues and eigenvectors of the matrix

$$B = \begin{bmatrix} 6 & 1 & 2 & 0 & 0 \\ 0 & 6 & 1 & 2 & 0 \\ 0 & 0 & 6 & 1 & 2 \\ 2 & 0 & 0 & 6 & 1 \\ 1 & 2 & 0 & 0 & 6 \end{bmatrix} .$$

Hint: We have $B = 6I_5 + C + 2C^2$ for some other matrix C about which you know a lot already.

Problem 7) (10 points)



A couple of years ago NASA researchers have announced the existence of bacteria for which some phosphor in the DNA is replaced by the element arsenic. Call p the number of phosphor bacteria and a the number of arsenic bacteria. Assume we have initially 100 phosphor bacteria, 0 arsenic bacteria and every night 90 percent of the p bacteria become arsenic bacteria and 80 percent of all arsenic bacteria become phosphorus bacteria. This system is written as $\vec{x}(t+1) = A\vec{x}(t)$ written out as

$$\begin{bmatrix} p(t+1) \\ a(t+1) \end{bmatrix} = \begin{bmatrix} 0.1 & 0.8 \\ 0.9 & 0.2 \end{bmatrix} \begin{bmatrix} p(t) \\ a(t) \end{bmatrix} .$$

a) (4 points) Find the eigenvalues and eigenvectors of A.

b) (4 points) Find a closed form solution of the system.

c) (2 points) The eigenvector v to the largest eigenvalue of A is the final equilibrium distribution. Find it and normalize it so that the sum of the entries is 100. The entries of v now give the final distribution in percentages.

Problem 8) (10 points)

Let $A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$.

a) (2 points) Compute A^2 and A^{-1}.

b) (2 points) Find the QR decomposition of A.

c) (2 points) Find the characteristic polynomial $f_A(\lambda)$ of A.

d) (2 points) If A is similar to a diagonal matrix D, find this matrix D, if not tell why there is none.

e) (2 points) Find a 2×2 matrix X so that $AX = A + A^2$.

a) $A^2 = \begin{bmatrix} 22 & 27 \\ 18 & 31 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 5 & -3 \\ -2 & 4 \end{bmatrix}/14$.

b) The QR factorization is (normalize the first vector to get u_1 then subtract $u_1 \cdot v_2$ times u_1 and normalize): $u_1 = [4,2]/\sqrt{20}$ and $w_2 = [3,5] - 22/20[4,2] = [-14,28]/10$ to get $[-1,2]/\sqrt{5}$.

$$A = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \cdot \begin{bmatrix} 2\sqrt{5} & 11/\sqrt{5} \\ 0 & 7/\sqrt{5} \end{bmatrix} .$$

c) $\lambda^2 - 9\lambda + 14$.

d) $D = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}$.

e) $X = (1 + A) = \begin{bmatrix} 5 & 3 \\ 2 & 6 \end{bmatrix}$ solves it.

Problem 9) (10 points)

a) (4 points) Find the determinant of the following matrix

$$\begin{bmatrix} 6 & 2 & 2 & 2 & 2 \\ 2 & 6 & 2 & 2 & 2 \\ 2 & 2 & 6 & 2 & 2 \\ 2 & 2 & 2 & 6 & 2 \\ 2 & 2 & 2 & 2 & 6 \end{bmatrix} .$$

Make sure to mention all tools you need to find the answer.

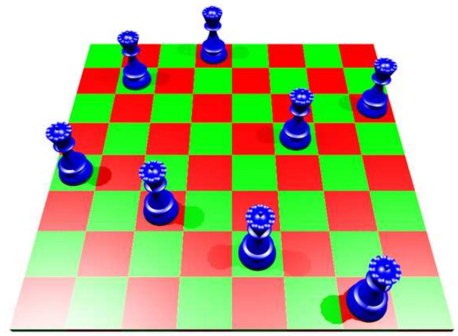
b) (3 points) Find the determinant of the following matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} .$$

Make sure to mention all tools you need to find the answer.

c) (3 points) The following matrix is a solution to the **eight queens puzzle**. Each entry 2 represents a **queen**. No queen can catch any other queen in chess. What is the determinant of this matrix? Make sure to mention all tools you need to find the answer.

$$\begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \end{bmatrix} .$$



Problem 10) (10 points)

Solve the following differential equations for which the initial position $f(0)$ and velocity $f'(0)$ is given:

a) (5 points)

$$f''(t) + 9f(t) = e^t, f(0) = 3, f'(0) = 0$$

b) (5 points)

$$f''(t) - 5f'(t) + 6f(t) = 2, f(0) = 3, f'(0) = -4$$

Problem 11) (10 points)

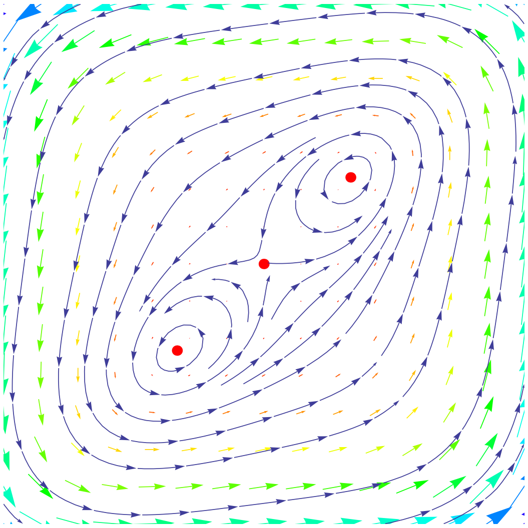
We analyze the following nonlinear dynamical system

$$\begin{aligned} \frac{d}{dt}x &= x - y^3 \\ \frac{d}{dt}y &= x^3 + y \end{aligned}$$

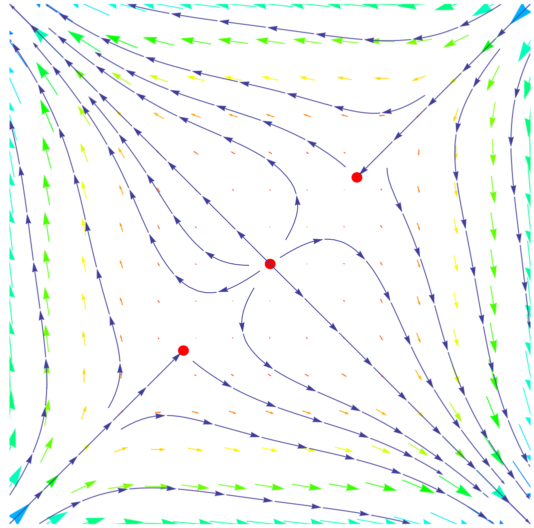
a) (3 points) Find the equations of the null-clines and find all the equilibrium points.

b) (4 points) Analyze the stability of the equilibrium point or equilibrium points.

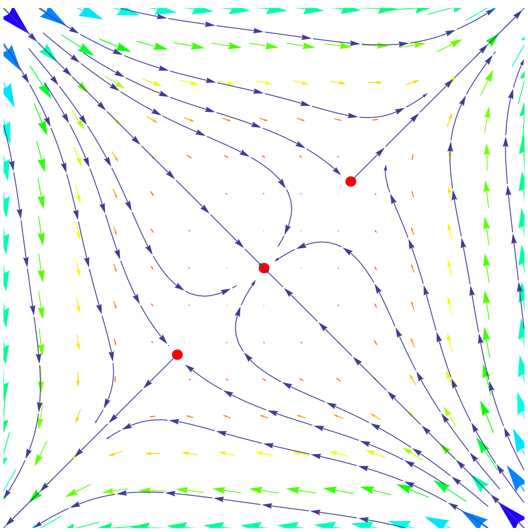
c) (3 points) Which of the four phase portraits A,B,C,D below belongs to the above system? Make sure that also here, you justify your answer, as always.



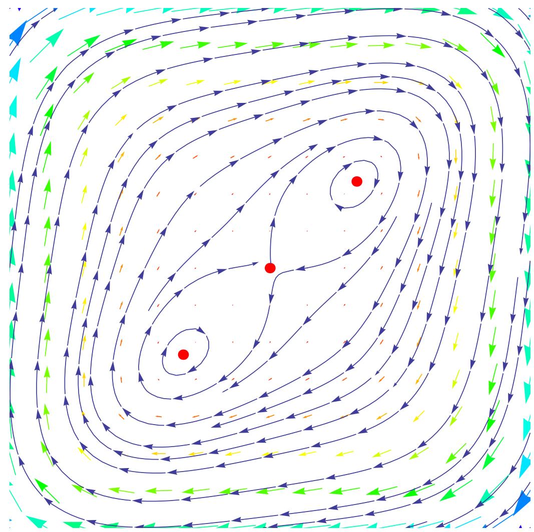
A



B



C



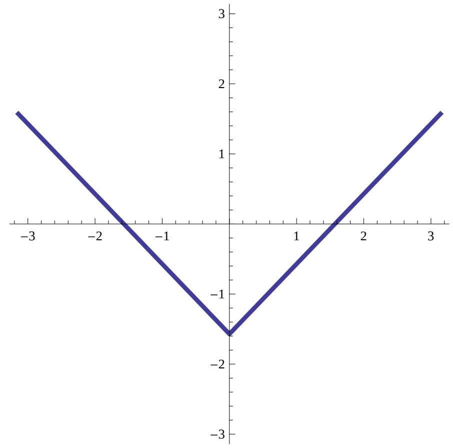
D

Problem 12) (10 points)

a) (7 points) Find the **Fourier series** of the function

$$f(x) = |x| - \frac{\pi}{2} .$$

The graph of the function f on $[-\pi, \pi]$ is displayed to the right.



b) (3 points) Use **Parseval's theorem** to find the value of the integral

$$\sqrt{\frac{1}{\pi}\int_{-\pi}^{\pi} (3\sin(5x) + 4\cos(12x) + 3\sin(x) + 4\cos(7x) + 5\cos(x) + 5\sin(2x))^2 \, dx}$$

without evaluating the integral directly.

Problem 13) (10 points)

The partial differential equation

$$u_{tt} = 9u_{xx} + \sin(1000t)$$

describes a **violin string** which is excited with a periodic force from the bow hair. The string is located on $[0, \pi]$ and has the initial condition $u(x, 0) = 2\sin(17x) + 5\sin(10x)$ and initial velocity $u_t(x, 0) = 5\sin(x) - 2\sin(3x)$. Find the motion $u(x, t)$ of the string.

a) (3 points) Find a special solution $u(x, t)$ which does not depend on x.

b) (5 points) Find a solution $u(x, t)$ of the homogeneous $u_{tt} = 9u_{xx}$ with the given initial condition.

c) (2 points) Write down the final solution of the problem.

Problem 14) (10 points)

About 10 years ago, we went for the record for the largest matrix which ever appeared in an exam, textbook etc. (see one of the older practice exams). Lets top the record and go for a 40 times 40 matrix: find the determinant of the following 40×40 matrix with 1600 matrix entries:

$$\begin{bmatrix}
3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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\end{bmatrix}$$