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- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) (20 points) True or False? No justifications are needed.

1) T F The matrix $\begin{bmatrix} 2 & -c \\ c & 3 \end{bmatrix}$ is always invertible for $c \in \mathbb{R}$.

Solution:
The determinant is $6 + c^2$.

2) T F The solutions of $f''' + f'' + 17f = e^t$ form a linear subspace of $C^\infty(\mathbb{R})$

Solution:
It is not a linear subspace because for example, it does not contain $f = 0$.

3) T F The solutions of $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ form a linear subspace of $\mathbb{R}^2$.

Solution:
The vector 0 is not in the set.

4) T F If A and B are 5×5 matrices, then $\text{rank}(A + B) = \text{rank}(A) + \text{rank}(B)$.

Solution:
Take for example invertible matrices. They have rank 5 which is maximal already.

5) T F Similar matrices have the same rank.

Solution:
Bring both matrices into Jordan normal form. The rank is the number of nonzero diagonal entries.

6) T F If A is a matrix which has orthonormal columns, then $\det(AA^T) = \det(A^TA)$.

Solution:
If A has less columns then rows, the right hand side has always determinant 1, the left hand side has always determinant 0. The statement would be true if A were a square matrix.

7) T **F** If A is a 2×2 matrix with $\det(A) < 1$, then the discrete dynamical system $\vec{x}(t+1) = A\vec{x}(t)$ has a stable origin.

Solution:
The absolute value of the eigenvalues matters. You can have a matrix with eigenvalues $2, -3$ for example, for which the determinant is -6 but this is not stable.

8) T **F** If A, B are $n \times n$ matrices, then $\det(2A + 3B) = 2^n \det(A) + 3^n \det(B)$.

Solution:
The determinant is not linear.

9) **T** F If A and B are 2×2 matrices with the same trace and the same determinant, then A and B have the same eigenvalues.

Solution:
The characteristic polynomial of 2×2 matrices depends only on trace and determinant.

10) T **F** If $A = QR$ is the QR decomposition obtained by Gram-Schmidt orthogonalization, then A and R have the same eigenvalues.

Solution:
Take a pure rotation. The eigenvalues of R are the diagonal entries of R which are 1. But the eigenvalues of the rotation are not 1.

11) T **F** If $\vec{x}^*$ is the least-squares solution of $A\vec{x} = \vec{b}$ then $\vec{b}$ is orthogonal to $A\vec{x}^*$.

Solution:
$A\vec{x}^*$ is equal to $\vec{b}$ by assumption and is not orthogonal to itself.

12) **T** F

If two matrices are symmetric and have the same eigenvalues (with the same algebraic multiplicities), then they are similar.

Solution:
Indeed, both matrices are then similar to the same diagonal matrix.

13) **T** F

Every system of linear equations has a least square solution.

Solution:
It is the projection onto the image of A. Note that if the kernel of A is not trivial, then there are many solutions.

14) T **F**

If a real 2×2 matrix A has i as an eigenvalue, it is orthogonal.

Solution:
Take $A = \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix}$, any nonorthogonal invertible matrix S and form $B = S^{-1}AS$. This is in general not invertible. For example $B = \begin{bmatrix} 18 & 25 \\ -13 & -18 \end{bmatrix}$ has eigenvalues $i, -i$.

15) **T** F

If $A = BCD$, where A, B, C, D are all 3×3 matrices and A is not invertible, then one of the matrices B, C, D are not invertible.

Solution:
If B, C, D were all invertible, the determinant of the product would be nonzero.

16) T **F**

If every eigenvalue λ of a matrix A satisfies $\mathrm{Re}(\lambda) < 1$, then $\vec{0}$ is an asymptotically stable equilibrium of the discrete dynamical system $\vec{x}(t+1) = A\vec{x}(t)$.

Solution:
It is the absolute value of the eigenvalues which matters.

17) T **F**

There is an $n \times n$ matrix A which has an eigenvalue λ of geometric multiplicity 0.

Solution:
By definition, the geometric multiplicity is at least 1.

18) $\boxed{\text{T}}$ $\boxed{\text{F}}$ If $f(x) = 3\cos(7x) + 4\sin(2004x) + 2$, then $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\, dx = 33$.

Solution:
The Parseval equality gives that the integral is $3^2 + 4^2 + (2 * \sqrt{2})^2 = 33$.

19) $\boxed{\text{T}}$ $\boxed{\text{F}}$ 7 is an eigenvalue of $T(f) = f'' + 7f' + 77f$ on the space $X = C^\infty(\mathbb{R})$ of smooth functions on the real line $\mathbb{R}$.

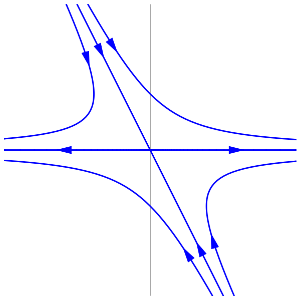
Solution:
Every real number is an eigenvalue of T.

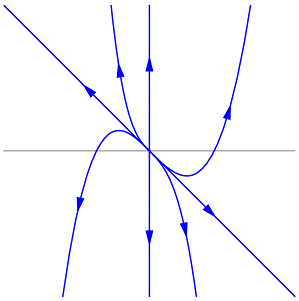
20) $\boxed{\text{T}}$ $\boxed{\text{F}}$ If $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \end{bmatrix} \right\}$ and $\vec{x} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ then $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

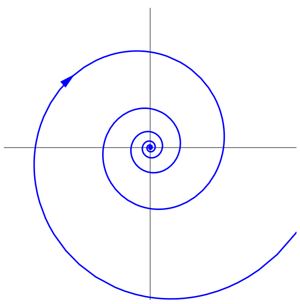
Solution:
We would have to have that -2 times the first eigenvector plus the second eigenvector is $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

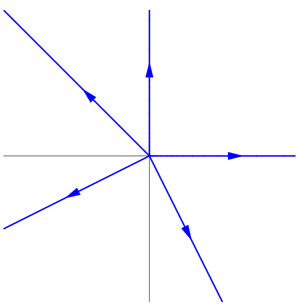
Problem 2) (10 points)

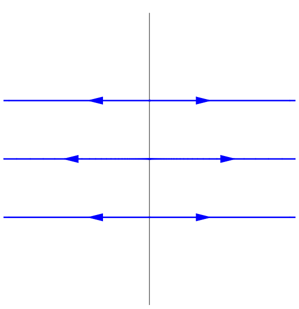
Pick the five of the dynamical system 1) - 9) which correspond to the phase portraits.











1. $\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}$

2. $\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \vec{x}$

3. $\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \vec{x}$

4. $\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \vec{x}$

5. $\frac{d\vec{x}}{dt} = \begin{bmatrix} -1 & 0 \\ -2 & -2 \end{bmatrix} \vec{x}$

6. $\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \vec{x}$

7. $\frac{d\vec{x}}{dt} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \vec{x}$

8. $\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 7 \\ -7 & 1 \end{bmatrix} \vec{x}$

9. $\frac{d\vec{x}}{dt} = \begin{bmatrix} -1 & 7 \\ -7 & -1 \end{bmatrix} \vec{x}$

Solution:
The solution code is 3,2,9,7,4.

- The first problem is an example, where one eigenvalue is positive, the other is negative. Note that systems 1-7 are upper or lower triangular matrices for which one can read of the eigenvalues directly.
- The second problem is an example, with two positive but different eigenvalues.
- The third example is an example with a complex eigenvalue where the real part is negative. Note that 8 and 9 are rotation-dilation matrices for which one can read of the eigenvalues directly too.
- The fourth phase portrait is an example with two same positive eigenvalues.
- The second example satisfies y'=0. This is system 4.

Problem 3) (10 points)

To match the dynamical systems to the left with the description to the right, fill in a)-e) in the boxes. No justifications are necessary.

a) $\frac{d}{dt}x = \sin(xy), \frac{d}{dt}y = x^2 + y$

b) $f_t = f_{xxxx}$

c) $\vec{x}(t+1) = A\vec{x}(t)$

d) $\frac{d}{dt}\vec{x} = A\vec{x}$

e) $f'' + f' + f = \sin(t)$

	Partial differential equation
	Linear system of ordinary differential equations
	nonlinear differential equation
	inhomogeneous linear differential equation
	discrete dynamical system.

To match the matrices to the left with the description to the right, distribute a)-e) in the boxes.

a) $\begin{bmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$

b) $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

c) $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$

e) $\begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}$

	skew symmetric matrix
	rotation
	reflection
	projection
	shear

Solution:
b,d,a,e,c
e,b,a,c,d

Problem 4) (10 points)

Let $A = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 2 & 5 \end{bmatrix}$.

a) (3 points) Find all eigenvalues of A with their algebraic multiplicities.

b) (3 points) Find the geometric multiplicities of each eigenvalue.

c) (2 points) Is A diagonalizable?

d) (2 points) What is the determinant of A^3?

Solution:
a) A is a partitioned matrix. The upper matrix is upper triangular, the lower matrix is lower triangular. The eigenvalues are $3, 4, 5$ with algebraic multiplicities $1, 1, 2$.
b) The geometric multiplicities are $1, 1, 1$. The eigenvalue 5 has geometric multiplicity 1.
c) No. If it were diagonalizable, then the geometric multiplicities had to be equal to the algebraic multiplicities.
d) The determinant of A is the product of the eigenvalues which is $25 \cdot 12 = 300$. The determinant of A^3 is 300^3.

Problem 5) (10 points)

Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

a) Find A^{-1}.

b) Solve $A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

c) Find the matrix of $T(\vec{x}) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \vec{x}$ with respect to the basis $\mathcal{B} = \{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \}$.

Solution:

a) $A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ 2 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$. b) $\vec{x} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$.

c) $S = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. We have $B = S^{-1}AS =$ $\begin{bmatrix} -1 & 1 & 0 \\ 2 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 2 & 3 & 0 \\ 1 & 1 & 3 \end{bmatrix}$.

Problem 6) (10 points)

You have only to solve 5 from the following 6 problems to have full credit. But you can attempt all of them.

a) (2 points) Find a 3×3 matrix A of rank 1 with no zero entries.

b) (2 points) Find a matrix A which has $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in the image of A.

c) (2 points) Find a 3×3 matrix A whose kernel is 2-dimensional.

d) (2 points) Find a 2×2 matrix A with different eigenvalues such that $A^2 - 3A + 2I_2$ is the zero matrix.

e) (2 points) Find a 2×2 matrix A for which A^{-1} and A^T have the same eigenvectors.

f) (2 points) Find a 3×3 matrix A such that every vector in $\mathbb{R}^3$ is an eigenvector of A with eigenvalue 3.

Solution:
Note that for all these questions, there are many different solutions.

a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

b) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.

c) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

d) The eigenvalues must be $2, 1$. Take the diagonal matrix with these entries: $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.

e) Take the identity matrix. Any symmetric matrix would work. Also any diagonal or any orthogonal matrix would work.

f) Take $3I_3$.

Problem 7) (10 points)

Let $A = \begin{bmatrix} 1 & 2 & -3 & 3 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & -1 & 1 \\ 0 & -2 & 3 & -4 \end{bmatrix}$.

a) (4 points) Find a basis of $\ker(A)$.

b) (2 points) Find the rank of A.

c) (1 point) Is there a vector $\vec{b}$ such that $A\vec{x} = \vec{b}$ has no solution?

d) (1 point) Is there a vector $\vec{b}$ such that $A\vec{x} = \vec{b}$ has exactly one solution?

e) (1 point) Is there a vector $\vec{b}$ such that $A\vec{x} = \vec{b}$ has infinitely many solutions?

f) (1 point) Find $\det(A)$.

Solution:

a) Row reduction gives $rref(A) = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The kernel is one dimensional and is spanned by $\begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$.

b) By the dimension formula, the rank is $4 - 1 = 3$.

c) Yes, the image is three dimensional. It is spanned by the first three column vectors of A.

d) No, the nontrivial kernel assures that there are either 0 or infinitely many solutions.

e) Yes, if $\vec{b}$ is in the image of A.

f) The determinant is 0 because the matrix is not invertible.

Problem 8) (10 points)

Let V be the plane spanned by $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$. Find the matrix of reflection at the plane V.

Solution:

First solution. First find an orthonormal basis in the plane. We can take $\vec{w}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}/3$, then do Gram-Schmidt $\vec{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}/3 = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}/3$. With $A = \begin{bmatrix} 1/3 & -2/3 \\ 2/3 & -1/3 \\ 2/3 & 2/3 \end{bmatrix}$, the projection onto the plane is $P = AA^T = \begin{bmatrix} 5 & 4 & -2 \\ 4 & 5 & 2 \\ -2 & 2 & 8 \end{bmatrix}/9$. The reflection matrix is $R\vec{x} = \vec{x} + 2(P\vec{x} - \vec{x}) = (2P - I_3)\vec{x}$. The matrix of the reflection is $\begin{bmatrix} 1 & 8 & -4 \\ 8 & 1 & 4 \\ -4 & 4 & 7 \end{bmatrix}/9$.

Second solution. The vector $\vec{w}_3 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$ is perpendicular to $\vec{w}_1$ and $\vec{w}_2$. The matrix $S = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix}/3$ is a coordinate change into the new basis. In that basis the transformation is $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. The original matrix is $SBS^{-1} = \begin{bmatrix} 1 & 8 & -4 \\ 8 & 1 & 4 \\ -4 & 4 & 7 \end{bmatrix}/9$.

Problem 9) (10 points)

a) (4 points) Find all the solutions of the differential equation $f' + 2f = e^{-2t}$.

b) (4 points) Find all the solutions of the differential equation $f'' + 4f' + 4f = e^{-2t}$.

c) (2 points) Find the kernel of $T(f) = f'' + 4f' + 4f$.

Solution:

a) We apply the formula for the inverse of $D + 2$ to get $f = (D + 2)^{-1}e^{-2t}$

$$f(t) = e^{-2t}(\int_0^t e^{2t}e^{-2t}\, dt + C) = Ce^{-2t} + te^{-2t} .$$

b) To invert $D^2 + 4D + 4 = (D + 2)^2$, we apply the formula for the inverse again $f(t) = (D+2)^{-1}(C_1e^{-2t}+te^{-2t}) = e^{-2t}(\int_0^t e^{2t}(C_1e^{-2t}+te^{-2t})\, dt+C_2) = e^{-2t}C_2+C_1te^{-2t}+\frac{t^2}{2}e^{-2t}$.

c) The homogeneous solutions of b) are spanned by e^{-2t} and te^{-2t}.

Problem 10) (10 points)

We analyze the nonlinear dynamical system

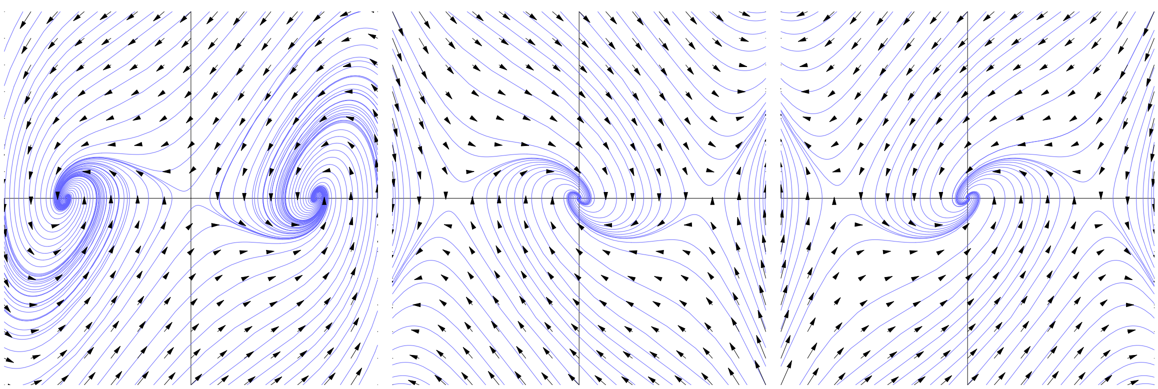
$$\begin{aligned} \frac{d}{dt}x &= y \\ \frac{d}{dt}y &= x^3 - x - y \end{aligned}$$

a) (2 points) Draw the nullclines, and indicate the direction of the field along the nullclines and inside the regions determined by the nullclines.

b) (2 points) Find all the equilibrium points.

c) (4 points) Analyze the stability of all the equilibrium points

d) (2 points) Which of the phase portraits A,B,C below belong to the above system?



A B C

Solution:
a) The nullclines are the line $y = 0$ as well as the curve $y = x^3 - x$.
b) The equilibrium points are the intersection of the nullclines which are $(-1, 0), (0, 0), (1, 0)$.
c) The Jacobian matrix is $J = \begin{bmatrix} 0 & 1 \\ 3x^2 - 1 & -1 \end{bmatrix}$. At the points $(-1, 0)$ and $(1, 0)$, it is $J = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$ which has eigenvalues $(-2, 1)$. Solutions are unstable those points. At the point $(0, 0)$ it is $J = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$ which has eigenvalues $(-1 \pm i\sqrt{3})/2$ which shows that this equilibrium point is stable. The phase portrait B is the correct solution.

Problem 11) (10 points)

Find the Fourier series of the function $f(x) = \cos(x) + \sin(2x) + x$ defined on $[-\pi, \pi]$. Show all computation steps.

Solution:
Note that $\cos(x)$ and $\sin(2x)$ are already part of the Fourier series. We can treat them separately. The Fourier series of $f(x) = x$ which is an odd function and has a pure sin series. The coefficients $b_n = \frac{2}{\pi}\int_0^\pi x\sin(nx)\ dx$ are computed using partial integration with $u = x, dv = \sin(nx)$: $\frac{2}{\pi}x(-\frac{\cos(nx)}{n})|_0^\pi + \frac{1}{n}\int_0^\pi \cos(nx)\ dx = -\frac{2}{n}\cos(n\pi) = \frac{2(-1)^{n+1}}{n}$. The solution is

$$f(x) = \cos(x) + \sin(2x) + \sum_{n=1}^\infty \frac{2(-1)^{n+1}}{n}\sin(nx)\ .$$

Problem 12) (10 points)

a) Find the solution of the heat equation $f_t = 3f_{xx}$ for which $f(x,0) = \sin(x) + \frac{1}{2}\sin(10x)$.

b) Find the solution of the wave equation $f_{tt} = 400f_{xx}$ for which $f(x,0) = \sin(x) + \frac{1}{2}\sin(10x)$ and $f_t(x,0) = \sin(20x)$.

Solution:
Note that all functions which appear are already the Fourier series. Each of the summands are already eigenfunctions of μD^2 rsp. c^2D^2. No need to compute Fourier coefficients. Note that in a), the constant μ is $\mu = 3$ and that in b) the constant c is $c = 20$.
a)

$$f(x,t) = \sin(x)e^{-3t} + e^{-300t}\frac{\sin(10x)}{2}$$

b) We note that $c = 20$ and the only n which matter are $n = 1, n = 10$ and $n = 20$. We have

$$f(x,t) = \cos((20\cdot 1)t)\sin(x) + \frac{\cos((20\cdot 10)t)}{2}\sin(10x) + \frac{\sin((20\cdot 20)t)}{20\cdot 20}\sin(20x)$$

or simplified

$$f(x,t) = \cos(20t)\sin(x) + \frac{\cos(200t)}{2}\sin(10x) + \frac{\sin(400t)}{400}\sin(20x)\ .$$

Problem 13) (10 points)

A vibrating string with friction is modeled by the **driven wave equation**

$$u_{tt} = u_{xx} - u \ .$$

Find the solution of this equation if $u(x,0) = 3\sin(5x)$ and initial speed $u_t(x,0) = \sin(11x)$. As usual, we work with $x \in [0,\pi]$.

Solution:
The Fourier decomposition of the initial wave position is $3\sin(5x)$, the function itself. The Fourier decomposion of the initial wave velocity is $\sin(11x)$ already. Note that the right hand side of the wave equation is of the form $D^2 - 1$ which has eigenvalues $-1-n^2$. The equation

$$u_{tt} = (-1-n^2)u$$

has the solutions $\cos(\sqrt{1+n^2}t)u(0) + \sin(\sqrt{1+n^2}t)u'(0)/\sqrt{1+n^2}$ because we know that the harmonic oscillator

$$u_{tt} = -c^2 u$$

has the solutions $\cos(ct)u(0) + \sin(ct)u'(0)/c$. The solution of the wave equation is

$$f(x,t) = 3\cos(\sqrt{5^2+1}t)\sin(5x) + \sin(\sqrt{11^2+1}t))\sin(11x)/\sqrt{11^2+1} \ .$$

Problem 14) (10 points)

If D is the differentiation operator $Df(x) = f'(x)$, we can define $e^D f(x) = \sum_{n=0}^{\infty} \frac{D^n f}{n!}$.

a) Verify that $e^{Dt} f(x)$ satisfies the transport equation $\frac{d}{dt} f = Df$.

b) Also $f(x+t)$ is a solution to the transport equation $f_t = Df$. It must therefore be the same as the solution $e^{Dt} f$ obtained in a). We have derived the equation

$$f(x+t) = e^{Dt} f \ .$$

It is known the name Taylor formula.

Solution:

a) Dfferentiation with respect to time gives $d/dt(e^{Dt}f) = D(e^{Dt}f)$ so that the function $f(x,t) = e^{Dt}f(x)$ solves the transport equation $u_t = Du = u_x$.
b) This is the Taylor formula

$$f(x+t) = f(x) + \frac{f'(x)t}{1!} + \frac{f''(x)t^2}{2!} + \frac{f'''(x)t^3}{3!} + \dots$$

known from single variable calculus. The English mathematician **Brook Taylor** lived from 1685 to 1731.