Name:

MWF9 George Boxer
MWF10 Omar Antolin
MWF10 Hector Pasten
MWF11 Oliver Knill
MWF12 Gabriel Bujokas
MWF12 Cheng-Chiang Tsai
TThu10 Simon Schieder
TThu11 Arul Shankar

- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (20 points) No justifications needed

1) T F A linear system with 2 equations and 3 unknowns has either infinitely many or no solutions.

2) T F If S is an invertible matrix which contains the vectors $\vec{v}_1, ..., \vec{v}_n$ as columns, then $\vec{v}_1, \dots, \vec{v}_n$ is a basis of $\mathbf{R}^n$.

3) T F If A, B are given $n \times n$ matrices, then the formula $(A-B)(A+B) = A^2 - B^2$ holds.

4) T F Suppose A is an $m \times n$ matrix, where $n < m$. If the rank of A is m, then there is a vector $y \in \mathbf{R}^m$ for which the system $Ax = y$ has no solutions.

5) T F The matrix $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 3 & 3 & 3 \end{bmatrix}$ is invertible.

6) T F The rank of an lower-triangular matrix equals the number of non-zero entries along the diagonal.

7) T F The row reduced echelon form of a 3×3 matrix of rank 2 is one of the following $\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

8) T F The matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is a shear.

9) T F For any matrix A, one has $\dim(\ker(A)) = \dim(\ker(\mathrm{rref}(A)))$.

10) T F If A is a square matrix for which $\ker(A)$ is included in $\mathrm{im}(A)$, then A is not invertible.

11) T F There exists an invertible 3×3 matrix, for which 7 of the 9 entries are π.

12) T F If $A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$, then $A^T A$ is the projection onto a line.

13) T F If A and B are $n \times n$ matrices, then AB is invertible if and only if both A and B are invertible.

14) T F There exist matrices A, B such that A has rank 4 and B has rank 7 and AB has rank 5.

15) T F There exist matrices A, B such that A has rank 2 and B has rank 7 and AB has rank 1.

16) T F If for an invertible matrix A one has $A^2 = A$, then $A = I_n$.

17) T F If an invertible matrix A satisfies $A^2 = I_2$, then $A = I_2$ or $A = -I_2$.

18) T F The matrix $\begin{bmatrix} c-1 & -1 \\ 2 & c+1 \end{bmatrix}$ is invertible for every real number c.

19) T F For 2×2 matrices A and B, if $AB = 0$, then either $A = 0$ or $B = 0$.

20) T F If T is a rotation in space with an angle $\pi/6$ around the z axes, then the linear transformation $S(x) = T(x) - x$ is invertible.

Problem 2) (10 points)

Match each of matrices with one of the geometric descriptions below. You don't have to give explanations.

Matrix	Enter A-H here.	Matrix	Enter A-H here.
a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$		e) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$	
b) $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$		f) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	
c) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$		g) $\begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	
d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$		h) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	

A) Shear along a plane.
B) Projection onto a plane.
C) Rotation around an axes.
D) Reflection at a point.
E) Projection onto a line.
F) Reflection at a plane.
G) Reflection at a line.
H) Identity transformation.

Problem 3) (10 points)

a) Write the matrix $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ as a product of a rotation and a dilation.

b) What is the length of the vector $\vec{v} = A^{100} e_1$, where e_1 is the first basis vector?

c) In which direction does the vector $\vec{v}$ point?

d) Find a matrix B such that $B^2 = A$.

Problem 4) (10 points)

Let A be a 3×3 matrix such that $A^2 = 0$. That is, the product of A with itself is the zero matrix.

a) Verify that $\text{Im}(A)$ is a subspace of $\ker(A)$.

b) Can $\text{ran}(A) = 2$? If yes, give an example.

c) Can $\text{ran}(A) = 1$? If yes, give an example.

d) Can $\text{ran}(A) = 0$? If yes, give an example.

Problem 5) (10 points)

Let b, c be arbitrary numbers. Consider the matrix $A = \begin{bmatrix} 0 & -1 & b \\ 1 & 0 & -c \\ -b & c & 0 \end{bmatrix}$.

a) Find $\text{rref}(A)$ and find a basis for the kernel and the image of A.

b) For which b, c is the kernel one dimensional?

c) Can the kernel be two dimensional?

Problem 6) (10 points)

Consider the matrix $A = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 3 & 0 & 0 & 0 \end{bmatrix}$.

a) Use a series of elementary Gauss-Jordan row operations to find the reduced row echelon form $\text{rref}(A)$ of A. Do only one elementary operations at each step.

b) Find the rank of A.

c) Find a basis for the image of A.

d) Find a basis for the kernel of A.

Problem 7) (10 points)

Let A be a 2×2 matrix and $S = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$. We know that $B = S^{-1}AS = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Find A^{2003}.

Problem 8) (10 points)

Let A be a 5×5 matrix. Suppose a finite number of elementary row operations reduces A to the following matrix $B = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$.

a) Find a basis of the kernel of A.

b) Suppose the elementary row operations used in reducing A to B are the following:

i) Add row 2 to row 3.
ii) Swap row 2 and row 4.
iii) Multiple row 4 by 1/2.
iv) Subtract row 1 from row 5.

Find a basis of the image of A.

Problem 9) (10 points)

a) Find a basis for the plane $x + 2y + z = 0$ in $\mathbf{R}^3$.

b) Find a 3×3 matrix which represents (with respect to the standard basis) a linear transformation with image the plane $x + 2y + z = 0$ and with the kernel the line $x = y = z$.

Problem 10) (10 points)

The set of 2×2 matrices form a linear space like vectors. One can add them, scale them and the zero matrix is the zero element. If you like you could stack the two column vectors onto each other to get a vector of length 4. But this is not really needed. Let T be the linear map from linear space X of 2×2 matrices to the real line which assigns to the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ its trace $T(A) = a + d$.

a) (3 points) What is the dimension of the image of T?
b) (3 points) What is the dimension of the kernel of T?
c) (3 points) Find an explicit nonzero matrix in the kernel of T.
d) (2 points) Is the transformation $T(A) = \det(A) = ad - bc$ a linear map from X to $\mathbb{R}$?