Name:

MWF9 George Boxer
MWF10 Omar Antolin
MWF10 Hector Pasten
MWF11 Oliver Knill
MWF12 Gabriel Bujokas
MWF12 Cheng-Chiang Tsai
TThu10 Simon Schieder
TThu11 Arul Shankar

- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1) T F

If u, v are two vectors in $\mathbf{R}^2$ which have length 1 and are perpendicular, then $u+v$ has length $\sqrt{2}$.

Solution:
Pythagoras.

2) T F

The rank of a matrix is always larger or equal than the dimension of the kernel.

Solution:
Take the 0 matrix, then the dimension of the kernel is n and the dimension of the image 0.

3) T F

There are two different 2×2 matrices A, B of rank 1 for which AB has rank 2.

Solution:
The rank can not get larger

4) T F

If A is a 1×1 matrix, then $\text{rref}(A)$ is either $[0]$ or $[1]$.

Solution:
Either there is a leading 1, then we have 1. Or there is none, and then $A = 0$.

5) T F

If a system of linear equations has at least 2 solutions, then it has at least 4 solutions.

Solution:
There are infinity.

6) T F

If S is the matrix containing a basis $\mathcal{B}$ as columns, and A represents a transformation in the standard basis and $Av = Sv$, then in the basis $\mathcal{B}$, the transformation has same matrix $B = A$.

Solution:
$B = S^{-1}AS = S = A$.

7) T F

The reflection A at the x axes is similar to the reflection B at the y axes. (Similar means that $B = S^{-1}AS$ for some S.)

Solution:
$T^2 = 1$ for a reflection but not for a shear

8) T F

A 3×3 matrix can have $\dim(\text{im}(A)) = \dim(\ker(A))$.

Solution:
The sum if odd but if both are the same, then the sum is even.

9) T F

The rank of a 17×13 matrix can be 14.

Solution:
The rank must be larger or equal then the minimum of n and m.

10) T F

If $\{v_1, v_2, v_3, v_4\ \}$ is a set of linearly independent vectors spanning a linear subspace V of $\mathbf{R}^9$, then $\dim(V) = 4$.

Solution:
By definition.

11) T F

For a 10×6 matrix A, the dimension of $\ker(A)$ is at least 4.

Solution:
It can be zero

12) T F

If A, B, and C are $n \times n$ matrices, then the property $A(BC) = (AB)C$ holds.

Solution:
This is a basic property of matrix multiplication

13) **T** F

Suppose A and B are $n \times n$ matrices. If A is invertible and B is not, then AB is not invertible.

Solution:
The matrix B has a kernel. If v is a vector in this kernel, then $ABv = 0$ too and AAB has a kernel.

14) **T** F

If a $n \times n$ matrix A has a nonzero vector v in the kernel, then it is not invertible.

Solution:
Yes, then the rank is n and the row reduction produces the identity matrix.

15) T **F**

If $A\vec{x} = \vec{0}$ has two linearly independent solutions, where A is a 5×5 matrix, then $\text{rank}(A) \leq 2$.

Solution:
Take the identity as a counter example.

16) **T** F

If A has the trivial kernel and $\vec{b}$ is in $\text{im}(A)$, then $A\vec{x} = \vec{b}$ has exactly one solution.

Solution:
There can be a kernel.

17) **T** F

If A and B are 2×2 matrices and AB is a rotation, then A and B are both invertible.

Solution:
Yes.

18) **T** F

If $\vec{v}$ is a nonzero vector in the kernel of A, then $\vec{v}$ is perpendicular to every row vector of A.

Solution:
This is what $A\vec{v} = \vec{0}$ means.

19) | T | F | For any two $n \times n$ matrices A and B, $(A+B)^2 = A^2 + 2AB + B^2$.

Solution:
No commutativity

20) | T | F | There is a 2×3 matrix A and a 3×2 matrix B such that AB is the identity and $BA = 0$.

Solution:
$AB = 1$ means that A, B both must have at least rank 2 and so kernel 1 or 0. This means that BA has a nullity of maximal 1.

Total

Problem 2) (10 points) No justifications are needed.

a) (5 points) Which of the following matrices can be put in row reduced echelon form in one single row reduction step (either swap, scale or subtract a row from an other row)?

Matrix	is in rref in one step	Matrix	is in rref in one step
$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$		$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	
$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$		$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$	
$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$		$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$	

b) (3 points) Which sets are linear spaces

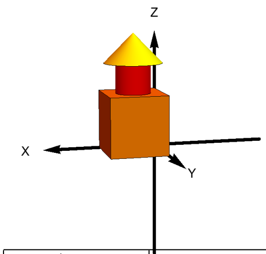
	The set of $\vec{x}$ which solve $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix} \cdot \vec{x} = 0$.
	The set of vectors $\vec{x}$ for which $x_1 + \cdots + x_n = 0$
	The image of the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}$.

c) (2 points) One of the following matrices is not invertible, which one?

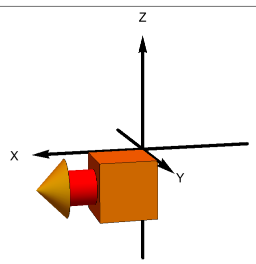
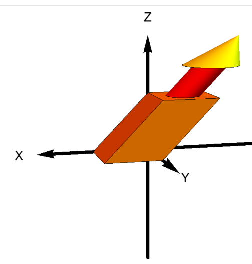
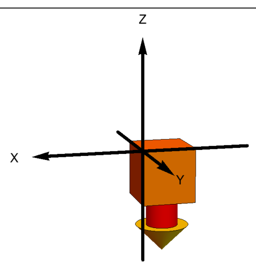
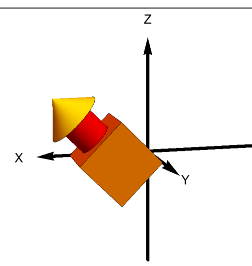
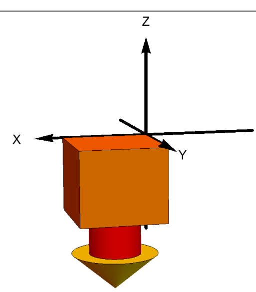
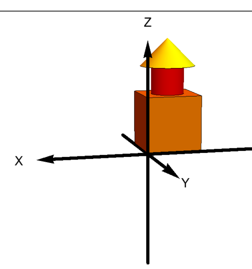
Matrix	not invertible
$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	
$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$	
$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}$	
$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	
$\begin{bmatrix} 5 & 4 & 3 & 2 \\ 4 & 3 & 2 & 0 \\ 3 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}$	

Solution:
a) yes yes
yes no
no no
b) All are linear spaces!
c) All except the third one are invertible.

Problem 3) (10 points) No justifications are necessary.



a) (6 points) Match the matrices with the action of the transformation which maps a shape using a transformation given by matrices $A - F$.

A-F		A-F	
			
			
			

$$A=\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad B=\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad C=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$D= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E= \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \quad F= \begin{bmatrix} 1.5 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & -1.5 \end{bmatrix}$$

b) (4 points) Which of the following matrices either perform a rotation dilation or a reflection dilation, which a projection dilation or shear dilation. (A shear dilation is a composition of a shear and a dilation).

Matrix	rotation dilation	reflection dilation	projection dilation	shear dilation
$\begin{bmatrix} 5 & 0 \\ 6 & 5 \end{bmatrix}$				
$\begin{bmatrix} 5 & 6 \\ 6 & -5 \end{bmatrix}$				

Solution:
a) CDBEFA
b) shear dilation, reflection dilation

Problem 4) (10 points)

Solve the following system of equations:

$$\begin{aligned} x+y+z+w &= 2 \\ x-y+z-w &= 4 \end{aligned}$$

Your result should give a parametrization of all the solutions.

Solution:
Row reduce the augmented matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & |2 \\ 1 & -1 & 1 & 1 & |4 \end{bmatrix} .$$

go get

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 1 & 1 & 3 \\ 0 & 1 & 0 & 0 & -1 \end{bmatrix} .$$

There are two leading 1 and two free variables s, t. Write down the equations $x + s + t = 3, y = -1, z = s, w = t$ and collect to get

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} .$$

Problem 5) (10 points)

Find a basis of the kernel and image of the 15 puzzle matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 0 \end{bmatrix} .$$

You might have played the 15 puzzle as a kid. The puzzle was invented by Noyes Palmer Chapman a postmaster in Canastota, New York as early as 1874. Copies of an improved puzzle made their way to Syracuse, New York to Watch Hill and finally to Hartford from where it was sold in Boston, where it got produced on a larger scale starting in 1879. A version with permuted 15-14 tiles was sold too. It is unsolvable, driving many mad.

Solution:
To find the image and kernel, row reduce the matrix. The fastest ist to subtract the row above in each case - watch out for that 0 on the right lower corner! - and end up with

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} .$$

There is one free variables. Write down the system again to get $x = 2, y = -2s, z = s, w = 0$ The image is spanned by the first, second and last column. The answer is

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 5 \\ 9 \\ 13 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 10 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \\ 12 \\ 0 \end{bmatrix} \right\} .$$

The kernel is spanned by $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$. The basis is

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\} .$$

Problem 6) (10 points)

a) (5 points) Invert the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

by using row reduction on an augmented 3×6 matrix.

b) (5 points) Find a basis for the linear space of vectors perpendicular to

$$v = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} .$$

Solution:
a) We row reduce the augmented matrix

$$A = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] .$$

The fastest is to subtract the second from the first and the third from the second, then swap the first and third. We get $A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$.

Even if it was possible that one do the row reduction in the head we wanted to see work done and not just the answer. This is common grading policy for all problems 4-9. Always show work, even if you can see the answer. We by purpose use exam problems with less complexity so that they are doable in the short time allocated.

b) To get the space, we find the kernel of the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}$. This is already in row reduced echelon form. There is one leading one and 4 free variables s, t, q, r. Write down the equation $x + 2s + 3t + 4u + 4v = 0, y = s, z = t, u = q, v = r$ and collect the same terms.

$$\mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} .$$

Problem 7) (10 points)

a) (5 points) Find the 3×3 matrix which reflects about the line through $v = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$.

b) (5 points) Find the 3×3 matrix which reflects about the plane $-x + 2z = 0$.

Solution:
a) $S = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}$, $S^{-1} = \begin{bmatrix} 1/3 & 0 & 2/3 \\ 2/3 & 0 & 1/3 \\ 0 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

$A = SBS^{-1} = \begin{bmatrix} -5/3 & 0 & -4/3 \\ 0 & -1 & 0 \\ 4/3 & 0 & 5/3 \end{bmatrix}$.

b) We can use the same S and B is the $-B$ from a). We get $A = SBS^{-1} = \begin{bmatrix} 5/3 & 0 & 4/3 \\ 0 & 1 & 0 \\ -4/3 & 0 & -5/3 \end{bmatrix}$.

Problem 8) (10 points)

Given the following matrices

$$A = \begin{bmatrix} -1 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

compute:

a) (2 points) AB

b) (2 points) BA

c) (2 points) ACB

d) (2 points) BCA

e) (2 point) C^2

Solution:
a) 2.
b) $\begin{bmatrix} -2 & 2 & 2 \\ -1 & 1 & 1 \\ -3 & 3 & 3 \end{bmatrix}$. c) [6].
d) Not defined.
e) $\begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}$.

Problem 9) (10 points)

Motivated by $\pi = 3.141...$ and $e = 2.718...$ we look at the vectors $x = \begin{bmatrix} 3 \\ 1 \\ 4 \\ 1 \end{bmatrix}$ and $y = \begin{bmatrix} 2 \\ 7 \\ 1 \\ 8 \end{bmatrix}$.

a) (4 points) Find the normalized vectors $X_i = x_i - E[x], Y_i = y_i - E[y]$, where $E[x] = (x_1 + x_2 + x_3 + x_4)/4$ and the dot product $X \cdot Y$, as well as the lengths $|X|, |Y|$. This corresponds up to a normalization to the correlation and standard deviation of the random variables X, Y in statistics.

b) (6 points) Find the correlation coefficient

$$\frac{X \cdot Y}{|X||Y|}$$

which we know to be the cosine between the angle between X and Y.

Solution:

a) $X = \begin{bmatrix} 3 \\ -5 \\ 7 \\ -5 \end{bmatrix} /4,$

$Y = \begin{bmatrix} -5 \\ 5 \\ -7 \\ 7 \end{bmatrix} /2,$

$X \cdot Y = -32/2.$

$|X| = \sqrt{27/4}, |Y| = \sqrt{37}.$

b) $X \cdot Y/(|X||Y|) = -31/(3\sqrt{111}).$