Name:

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- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) (20 points) True or False? No justifications are needed.

1)	T	F	The formula for the projection onto the image of a matrix A is $A(A^TA)^{-1}A^T$.
2)	T	F	Let A and B be two 3×3 matrices. Then A and B are similar if and only if they have the same eigenvalues.
3)	T	F	Let A be any $n \times n$ matrix. If $\det(A^TA) = 1$ or $\det(A^TA) = -1$, then A is an orthogonal matrix.
4)	T	F	A matrix which is both symmetric $A^T = A$ and skew-symmetric $A^T = -A$ is orthogonal.
5)	T	F	The trace of a matrix A does not change under row reduction.
6)	T	F	The eigenspace to the eigenvalue 0 of a matrix A does not change under row reduction.
7)	T	F	If A is any matrix of a rotation around a line in space, then $\det(A - I) = 0$.
8)	T	F	If A is diagonalizable and B is diagonalizable, then $A + B$ is always diagonalizable.
9)	T	F	There is a recursion $x_{n+1} = ax_n + bx_{n-1}$ for which $x_n = \sqrt{n}$ for all n.
10)	T	F	The product of two reflections at a line in the plane is always a reflection at a line.
11)	T	F	The characteristic polynomial of A is the same as the characteristic polynomial of A^{-1}.
12)	T	F	For any 3×3 matrix, we have $\det(A^4) = \det(A)^{12}$.
13)	T	F	The determinant of a matrix is equal to the sum of the eigenvalues.
14)	T	F	There is a reflection at a line in the plane for which the determinant is equal to 1.
15)	T	F	The matrix $A = \begin{bmatrix} 100 & 0 \\ 10000 & 1000 \end{bmatrix}$ is similar to $B = \begin{bmatrix} 1000 & 100000 \\ 0 & 100 \end{bmatrix}$.
16)	T	F	If two 2×2 matrices A and B are similar, they have the same trace and determinant.
17)	T	F	If y is in $\mathrm{im}(A)$ then the least square solution to $Ax = y$ is an actual solution to $Ax = y$.
18)	T	F	It is possible that $\mathrm{tr}(A^n)$ and $\mathrm{tr}(A^{-n})$ both grow exponentially.
19)	T	F	If a symmetric matrix Q is orthogonal, then Q is diagonal.
20)	T	F	If $A = QR$ is the QR decomposition of a square matrix, then the eigenvalues of A are the diagonal entries of R.

Total

Problem 2) (10 points)

a) (4 points) Which of the following matrices are diagonalizable. No justifications are necessary.

1) $\begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ ☐ 2) $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ ☐

3) $\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ ☐ 4) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ ☐

b) (6 points) No justifications are necessary. Check the boxes, for which it is possible to find an example in the class of transformations in three dimensional space indicated to the left: (In this problem we also consider multiplication by zero as a dilation. In other words, also the zero matrix is considered a dilation matrix.)

transformation $T(x) = Ax$	$\det(A) = 1$	$\det(A) = -1$	$\det(A) = 0$	$\det(A) = -2$
orthogonal rotation				
reflection at a line				
projection onto line				
standard shear at x axes				
dilation				
rotation dilation				

Problem 3) (10 points)

Define $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}$.

a) (4 points) Find the eigenvectors and the geometric multiplicities of the eigenvalues of A.

b) (3 points) Find the algebraic multiplicities of the eigenvalues.

c) (3 points) Find the characteristic polynomial $f_A(\lambda)$ of A.

Problem 4) (10 points)

Find the function $y = f(x) = ax^2 + bx^3$, which best fits the data

x	y
-1	1
1	3
0	10

Problem 5) (10 points)

a) (4 points) Find all the eigenvalues λ_1, λ_2 and eigenvectors v_1, v_2 of the matrix

$$A = \begin{bmatrix} 7 & 1 \\ 4 & 4 \end{bmatrix}.$$

b) (3 points) Find a formula for the characteristic polynomial $f_{A^n}(\lambda)$ of A^n, where n is a positive integer.

c) (3 points) What is $A^n \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ for a general positive integer n?

Problem 6) (10 points)

Find the point P on the plane V spanned by the two vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

closest to the point $b = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 1 \end{bmatrix}$.

Problem 7) (10 points)

a) (3 points) Find the determinant of the matrix

$$\begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}$$

b) (3 points) Find the determinant of the matrix

$$\begin{bmatrix} 3 & 1 & 1 & 2 & 2 & 2 \\ 1 & 0 & 1 & 2 & 2 & 2 \\ 1 & 0 & 3 & 2 & 2 & 2 \\ 0 & 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 & 1 & 4 \end{bmatrix}$$

c) (4 points) Find the determinant of

$$\begin{bmatrix} 3 & 2 & 0 & 0 & 0 & 0 \\ 3 & 3 & 2 & 0 & 0 & 0 \\ 3 & 3 & 3 & 2 & 0 & 0 \\ 3 & 3 & 3 & 3 & 2 & 0 \\ 3 & 3 & 3 & 3 & 3 & 2 \\ 3 & 3 & 3 & 3 & 3 & 3 \end{bmatrix}$$

Problem 8) (10 points)

The recursion

$$x_n = 2x_{n-1} - 2x_{n-2}$$

with $x_0 = 5, x_1 = 2$ can be rewritten as

$$\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = A^n \begin{bmatrix} x_1 \\ x_0 \end{bmatrix}$$

with $A = \begin{bmatrix} 2 & -2 \\ 1 & 0 \end{bmatrix}$.

a) (3 points) Find the eigenvalues and eigenvectors of A.

b) (4 points) Find a closed formula for x_n.

c) (3 points) Give an argument which verifies that A is similar to the rotation dilation matrix

$$B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} .$$

Note. You can leave powers of a complex number as they are. A term $(3+4i)^n$ for example does not have to be simplified further.

Problem 9) (10 points)

Let A be the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} .$$

a) (6 points) Find an orthonormal basis for the image of A.

b) (4 points) Find matrices B and C so that $A = BC$ and B is orthogonal and C is upper triangular.

Problem 10) (10 points)

a) Find all the eigenvalues λ_1, λ_2 and λ_3 of the matrix $A = \begin{bmatrix} 1 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

b) Find a formula for $\mathrm{tr}(A^n)$.