

1. The verification that $\cos(nx), \sin(nx), 1/\sqrt{2}$ form an orthonormal family is an integration computation, when using the identities provided. For example, $\langle \cos(nx), \sin(mx) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(nx) \sin(mx) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos((n-m)x) - \cos((n+m)x) dx$ which is equal to 1 if $n = m$ and equal to 0 if $n \neq m$. The computations can be abbreviated by noting that integrating an odd 2π -periodic function over $[-\pi, \pi]$ is zero because the integral on $[0, \pi]$ cancels with the integral on $[-\pi, 0]$.

2. First get the Fourier series of the function $f(x) = |x|$. This is an **even function** so that it has a cos series. We compute

$$a_0 = \langle f, 1/\sqrt{2} \rangle = \frac{2}{\pi} \int_0^{\pi} x \frac{1}{\sqrt{2}} dx = \frac{\pi\sqrt{2}}{2}.$$

$$a_n = \langle f, \cos(nx) \rangle = \frac{2}{\pi} \int_0^{\pi} x \cos(nx) dx = \frac{2}{\pi} \left[\frac{\cos(n\pi) - 1}{n^2} \right].$$

The Fourier coefficients of $f(x) = 5 - |2x|$ are given as follows:

$$a_0 = \frac{3\pi\sqrt{2}}{2} + 5\sqrt{2}$$

and

$$a_n = -\frac{4}{\pi} \left[\frac{\cos(n\pi) - 1}{n^2} \right].$$

3. The Fourier series of $4\cos^2(3x) + 5\sin^2(11x) + 90$ is with

$$\cos^2(3x) = \frac{1 + \cos(6x)}{2}$$

$$\sin^2(11x) = \frac{1 - \cos(22x)}{2}$$

given as $\boxed{4\cos(6x)/2 - 5\cos(22x)/2 + 9/2 + 90}$. All Fourier coefficients are zero except a_0 and a_6 and a_{22} .

4. To find the Fourier series of the function $f(x) = |\sin(x)|$, we first note that this is an **even function** so that it has a cos-series. If we integrate from 0 to π and multiply the result by 2, we can take the function $\sin(x)$ instead of $|\sin(x)|$ so that

$$a_0 = \frac{2}{\pi} \int_0^{\pi} \sin(x)/\sqrt{2} dx = \frac{2\sqrt{2}}{\pi}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} \sin(x) \cos(nx) dx = \frac{4}{\pi} \frac{1}{1 - n^2}$$

for even n and $a_n = 0$ for odd n .

To do the integral, use the trigonometric identities $2\sin(x)\cos(nx) = \sin(x+nx) + \sin(x-nx)$. We have

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi} \left(\frac{\cos(2x)}{2^2 - 1} + \frac{\cos(4x)}{4^2 - 1} + \frac{\cos(6x)}{6^2 - 1} + \dots \right).$$

5. The square of the length of the function $f(x)$ is 1. The Parseval identity shows that

$$1 = a_0^2 + \sum_{n=1}^{\infty} a_n^2 = (\sqrt{2}\frac{2}{\pi})^2 + \frac{16}{\pi^2} \left[\frac{1}{(2^2 - 1)^2} + \frac{1}{(4^2 - 1)^2} + \frac{1}{(6^2 - 1)^2} + \dots \right]$$

so that the sum is $\boxed{\pi^2/16 - 1/2}$.

6. To solve the heat equation $f_t = 5f_{xx}$ on $[0, \pi]$ with the initial condition $f(x, 0) = \max|\cos(x)|, 0$, we make a Fourier expansion

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

of the later function and the get solution

$$f(x, t) = \sum_{n=1}^{\infty} b_n e^{-5n^2 t} \sin(nx).$$

Now to the Fourier coefficients:

$$b_n = \frac{2}{\pi} \int_0^{\pi/2} \cos(x) \sin(nx) dx$$

We use the trig identity

$$\cos(x) \sin(ny) = \sin((n+1)x) + \sin((n-1)x)$$

to solve these integrals:

$$b_n = \frac{2}{\pi} \left[\frac{(1 - \cos((n+1)\pi/2))}{(n+1)} + \frac{(1 - \cos((n-1)\pi/2))}{(n-1)} \right].$$

7. The operator $D^4 + D^2$ has the eigenvectors $\sin(nx)$ with eigenvalues $n^4 - n^2$. With initial condition $f(x) = b_n \sin(nx)$ we have the solution $b_n \sin(nx) e^{(n^4 - n^2)t}$. The function x^3 has the Fourier coefficients $\frac{2}{n^3}(6 - n^2\pi^2)(-1)^n$.
8. Because the initial condition is zero on the interval $[\pi/2, \pi]$, we have to integrate from 0 to $\pi/2$ only. The Fourier coefficients of the function $g(x)$ can be computed using one of the trigonometric identities you find on the first page of the handout:

$$\frac{2}{\pi} \int_0^{\pi/2} \sin(2x) \sin(nx) dx = \frac{-4}{\pi(n^2 - 4)} \sin(n\pi/2).$$

The Fourier series of the initial position $f(x) = 0$ of the string is equal to zero by assumption. The solution of the wave equation is

$$f(x, t) = \sum_{n=1}^{\infty} \frac{-4}{\pi(n^2 - 4)} \sin(n\pi/2) \sin(nx) \sin(nt) \frac{1}{n}.$$

The solution also exists for $n = 2$, where it is $1/2$ (which can best be seen by evaluating the original integral $\frac{2}{\pi} \int_0^{\pi/2} \sin^2(2x) dx = 1/2$).

9. The general solution of the homogeneous equation with the function at rest initially is $u_h(t, x) = \sum_n b_n \sin(nx) \cos(nt) = 4 \cos(5t) \sin(5x) + 10 \cos(6t) \sin(6x)$. A particular solution which is zero at 0 is $u_p(t, x) = -\cos(t) - \cos(3t)/9 + (1 - 1/9)$. Now fix the Fourier coefficients. We end up with

$$u(t, x) = 4 \sin(5x) \cos(5t) + 10 \sin(6x) \cos(6t) - \cos(t) - \cos(3t)/9 + 8/9$$

which is the solution to the differential equation.

10. a) If the function is $\text{sign}(xy)$, we have

$$b_{nm} = \frac{4}{\pi^2} \int_0^\pi \int_0^\pi \sin(nx) \sin(my) dy dx$$

which is

$$\frac{4}{\pi^2} \left(-\frac{\cos(nx)}{n} \Big|_0^\pi \right) \left(-\frac{\cos(my)}{m} \Big|_0^\pi \right) = \frac{16}{\pi^2} \frac{1}{nm} .$$

The Fourier coefficients are $\boxed{\frac{16}{nm\pi^2}}$ if n, m are both odd and zero else.

- b) Since every initial condition $u = b_{nm} \sin(nx) \sin(my)$ satisfies the ordinary differential equation $u_t = (-n^2 - m^2)u$ with solution $u(t) = e^{-n^2 - m^2} u(0) = e^{-n^2 - m^2} b_{nm} \sin(nx) \sin(my)$. we can add up a linear combination of such solutions and get

$$u(x, y, t) = \sum_{n,m=1}^{\infty} b_{nm} e^{-(n^2+m^2)t} \sin(nx) \sin(my) .$$

With the Fourier coefficients computed in part a), we have the final answer

$$\boxed{u(x, y, t) = \sum_{n,m=\text{odd}}^{\infty} \frac{16e^{-(n^2+m^2)t}}{\pi^2 nm} \sin(nx) \sin(my)} .$$