MWF 9 Fabian Haiden
MWF 10 Ziliang Che
MWF 10 Jeremy Hahn
MWF 11 Rosalie Belanger-Rioux
MWF 11 Yu-Wen Hsu
MWF 12 Peter Garfield
TThu 10 Oliver Knill
TThu 11:30 Alex Perry
TThu 11:30 Rong Zhou

- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) (20 points) True or False? No justifications are needed.

1) T F If a linear system $Ax = 0$ has at least one solution, then the system $Ax = b$ has at least one solution for all b.

2) T F If A is an orthogonal matrix, then all matrix entries A_{ij} satisfy $|A_{ij}| \leq 1$ for all i, j.

3) T F The transformation $T(f)(x) = f(x^2) - 23f(x)$ is linear on the space of all polynomials.

4) T F If a smooth function f on $[-\pi, \pi]$ has a sin-Fourier expansion then it satisfies $\int_{-\pi}^{\pi} f(x)\, dx = 0$.

5) T F The characteristic polynomials of two real $n \times n$ matrices A, B satisfy $f_A(\lambda) + f_B(\lambda) = f_{A+B}(\lambda)$.

6) T F The function $f(t) = 23e^{10t}$ is an eigenfunction with eigenvalue 23 of the linear operator $T = D$, where $Df = f'$ is the differentiation operator on $C^\infty(\mathbf{R})$.

7) T F The matrix $(A^{23})(A^{23})^T$ is diagonalizable, if A is a real $n \times n$ matrix.

8) T F The initial value problem $f''(x) + 23f'(x) + 10f(x) = x + e^x$, $f'''(0) = 0$ has exactly one solution.

9) T F The transformation $T(f)(x) = \sin(x)f(\sin(x))$ is a linear transformation on the space $X = C^\infty(\mathbf{R})$ of smooth functions on the real line.

10) T F The set X of smooth functions $f(x, t)$ of two variables which satisfy the partial differential equation $f_{ttt} - f_{xxx} = f_x$ is a linear space.

11) T F If A is 23×23 matrix of rank 23, then it has an eigenvalue 0.

12) T F The vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ has the $\mathcal{B} = \{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}\}$-coordinates $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

13) T F If all the geometric multiplicities of the eigenvalues of a matrix are equal to the algebraic multiplicities, then the matrix is diagonalizable.

14) T F For a differential equation $\frac{d}{dt}x = f(x, y), \frac{d}{dt}y = g(x, y)$, every equilibrium point is an intersection of two nullclines.

15) T F If $z = 2i$ then $\sqrt{z} = 1 + i$ or $1 - i$.

16) T F The determinant and trace of a 2×2 matrix A always satisfy the inequality $\text{tr}(A) \leq \det(A)$.

17) T F The QR decomposition of an upper triangular matrix A with positive diagonal entries is $A = QR$, where $R = A$ and $Q = 1_n$.

18) T F If the trace and the determinant of a 2×2 matrix A are both zero, then A is the zero matrix.

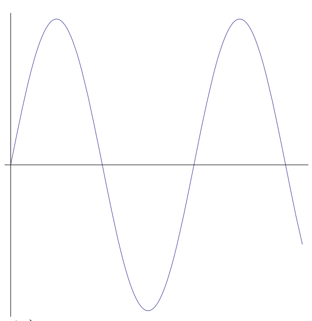
19) T F The discrete dynamical system $x(t+1) = x(t) + 23\, x(t-1)$ has the property that $|x(t)| \to \infty$ for all nonzero initial conditions $(x(0), x(1))$.

20) T F $||\sin(x) + \cos(23x)|| = \sqrt{2}$, where $||f|| = \sqrt{\langle f, f \rangle}$ is the length of the function f with respect to the inner product $\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)\, dx$.

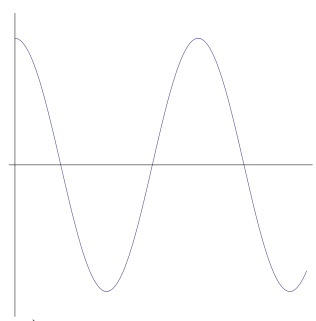
Problem 2) (10 points)

No justifications are needed in this problem. Match the equations with the solution graphs $f(x)$. Note that the graphs are not necessarily to scale. Enter five of the six choices A,B,C,D,E,F here:

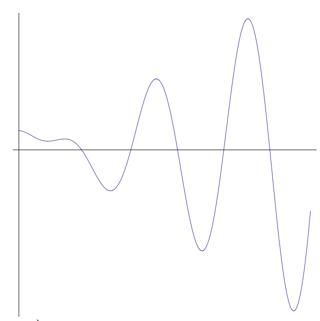
	$f''(t) + f(t) = \sin(t), f(0) = 1, f'(0) = 0$
	$f'(t) = -f(t), f(0) = 1$
	$f''(t) = -1, f(0) = 0, f'(0) = 1$
	$f''(t) = -\sin(t), f(0) = 0, f'(0) = 1$
	$f'(t) + f(t) = t, f(0) = 1$



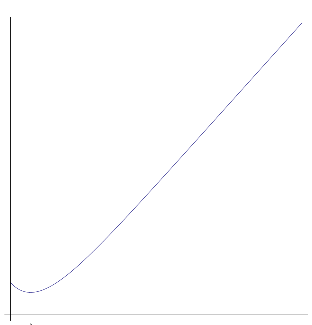
A)



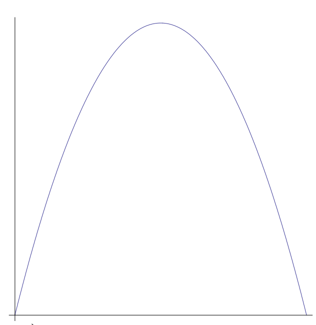
B)



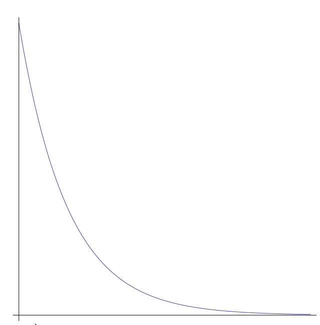
C)



D)



E)



F)

Problem 3) (10 points)

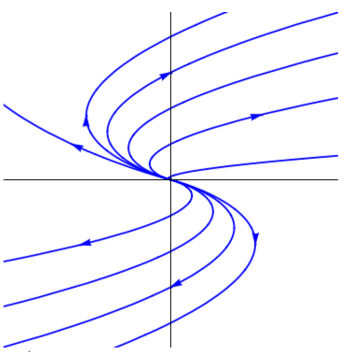
No justifications are needed in this problem. Check all boxes which apply except in the first column where you have to enter your choices of A)-F). We abbreviate the term "asymptotically stable" with "stable" and "diagonalizable" means diagonalizable over the complex numbers.

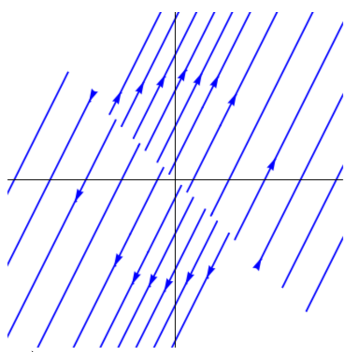
matrix A	phase A)-F)	$\frac{d}{dt}x = Ax$ stable	$x(t+1) = Ax(t)$ stable	A diagonalizable
$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$				
$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$				
$\begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}$				
$\begin{bmatrix} 0 & 2 \\ -2 & -1 \end{bmatrix}$				
$\begin{bmatrix} 3/4 & 4 \\ 0 & 2/4 \end{bmatrix}$				

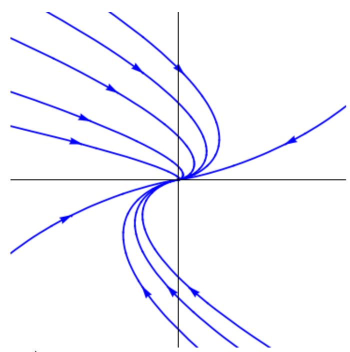
Below are 6 portraits for the continuous dynamical system

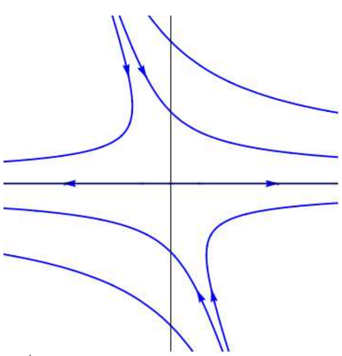
$$\frac{d}{dt}x = Ax \ .$$

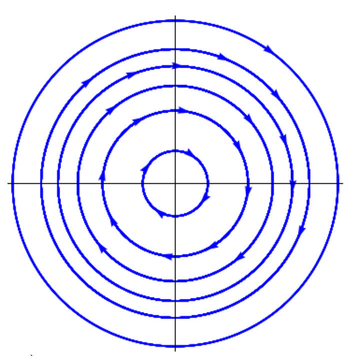
Please enter 5 of the 6 letters A-F in the table above.

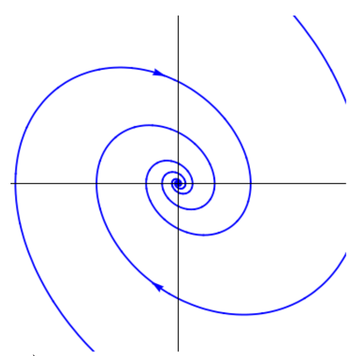

A)


B)


C)


D)


E)


F)

Problem 4) (10 points)

Find all the solutions of the system of linear equations for the variables x, y, z, u.

$$\left| \begin{array}{ccccccccc} x & - & y & & & & & = 4 \\ & & y & - & z & & & = 5 \\ & & & & z & + & u & = 7 \end{array} \right|$$

Problem 5) (10 points)

Find all the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 23001 & 3 & 5 & 7 & 9 & 11 \\ 1 & 23003 & 5 & 7 & 9 & 11 \\ 1 & 3 & 23005 & 7 & 9 & 11 \\ 1 & 3 & 5 & 23007 & 9 & 11 \\ 1 & 3 & 5 & 7 & 23009 & 11 \\ 1 & 3 & 5 & 7 & 9 & 23011 \end{bmatrix} .$$

As usual, document all your reasoning.

Problem 6) (10 points)

a) (3 points) Find the 3×3 matrix A belongs to the linear transformation which reflects a vector at the x axes.

b) (4 points) Find the 3×3 matrix B belongs to the linear transformation which projects onto the axes $x = y = z$.

c) (3 points) Find the matrix

$$AB - BA ,$$

the so called **commutator** of A and B.

Problem 7) (10 points)

Find the function $z = axy + bx^2$ which best fits the data

x	y	z
1	1	4
1	2	6
1	0	8

Problem 8) (10 points)

Solve the difference equation

$$x_{n+1} - x_n = 5y_n$$
$$y_{n+1} - y_n = 5x_n$$

with initial condition $x_0 = 5, y_0 = 7$.

Problem 9) (10 points)

Find the determinants of the following matrices:

a)

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \end{bmatrix}$$

b)

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 5 & 0 & 0 \end{bmatrix}$$

c)

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 1 & 1 & 6 \end{bmatrix}$$

Problem 10) (10 points)

Find the general solutions for the following differential equations:

a) (3 points) $f''(t) = t^2 + 3$.

b) (3 points) $f''(t) + f(t) = t^2 + 3$

c) (4 points) $f''(t) + 6f'(t) + 9f(t) = 2e^{-3t}$

Problem 11) (10 points)

We analyze the nonlinear dynamical system

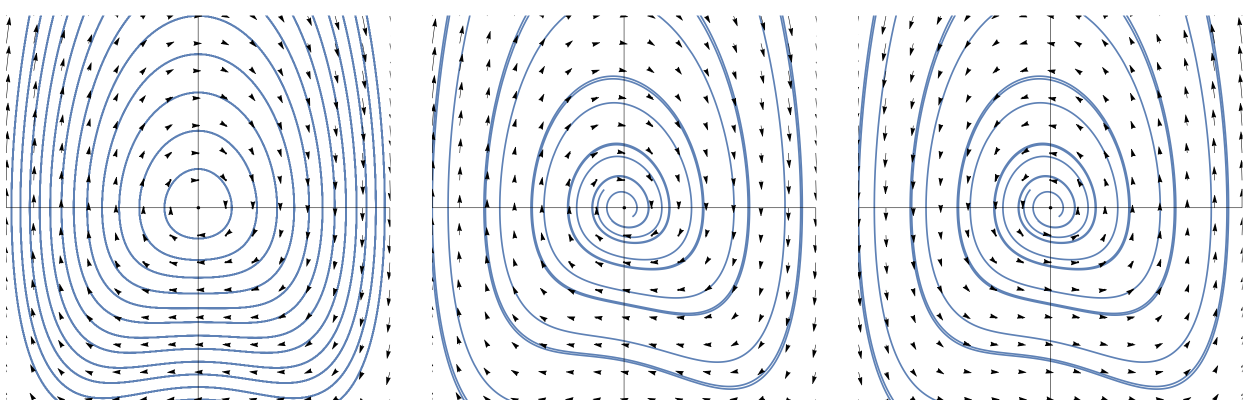
$$\begin{aligned} \frac{d}{dt}x &= y \\ \frac{d}{dt}y &= -x - xy - 2x^3 \end{aligned}$$

It is a variant of a **van der Pool oscillator**.

a) (3 points) Find the equations of the nullclines and find all the equilibrium points.

b) (4 points) Analyze the stability of all the equilibrium points.

c) (3 points) Which of the phase portraits A,B,C below belongs to the above system?



A B C

Problem 12) (10 points)

a) (5 points) Find the Fourier series of the function $f(x) = \begin{bmatrix} 1 & , |x| < \pi/4 \\ 0 & , |x| \geq \pi/4 \end{bmatrix}$. You do not have to simplify terms which look like $\sin(n\pi/3)$ or similar.

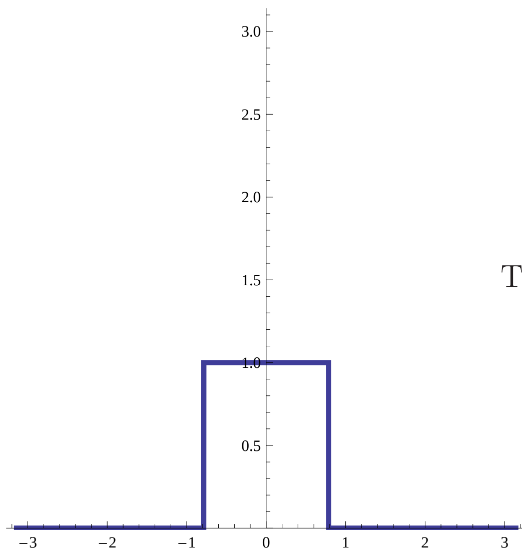
b) (5 points) The function $g(x) = x^3 - \pi^2 x$ has the Fourier series

$$g(x) = \sum_{n=1}^{\infty} \frac{12(-1)^n}{n^3} \sin(nx) .$$

What is

$$\sum_{n=1}^{\infty} \frac{1}{n^6} ?$$

This number is called $\zeta(6)$, the value of the **Riemann Zeta function** at 6.



The function $f(x)$ in problem 12a).

Problem 13) (10 points)

The partial differential equation

$$u_t = u_{xx} + bu = (D^2 + b)u$$

is a model for a **reaction diffusion process**, where a thermal process produces additional heat bu proportional to the given heat u. It could model rubbing a match at a matchbox.

Picture: Sean Oughton, Department of Mathematics, University of Waikato.

a) (3 points) Show that $\sin(nx)$ is an eigenvector=eigenfunction of the operator $T = D^2 + b$ in the PDE $u_t = T(u)$. What is the eigenvalue?

b) (2 points) Solve the system for initial condition $u(x, 0) = \sin(3x)+2\sin(5x)$ in the case $b = 1$.

c) (2 points) Solve the system for initial condition $u(x, 0) = \sin(3x) + 2\sin(5x)$ and general b.

d) (3 points) For large enough b, the heat production overcomes the dissipation. Assuming still the initial condition $u(x, 0) = \sin(3x) + 2\sin(5x)$, find the threshold value b_0 so that for $b > b_0$, the temperature $u(x, t)$ grows indefinitely and the match lights up.

Problem 14) (10 points)

A symmatric $n \times n$ matrix A has the QR decomposition $A = QR$.

a) (2 points) Verify that the matrix $B = RQ$ has the same eigenvalues than A.

b) (2 points) Is the map T which assigns to A the matrix B a linear map?

c) (2 points) What happens if T is applied to a diagonal matrix?

d) (4 points) An important method to compute the eigenvalues of A is to iterate the map T. This leads to matrices $A_1, A_2, ...$ which converge to a diagonal matrix, a fact which you can take for granted here. Demonstrate that this process also produces an orthonormal set of eigenvectors of A.

This QR method to find eigenvalues and eigenvectors has been discovered independently by the mathematicians **J.G.F. Francis** and **Vera N. Kublonovskaya** in 1961. Its a simple method compaed to finding roots of characteristic polynomials and finding kernels for each root.