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- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) (20 points) True or False? No justifications are needed.

1) T **F** If A is an invertible matrix, then A^{-1} and A^T have the same eigenvalues.

Solution:
This is already wrong for the 1×1 matrix $A = 2$.

2) **T** F For all $n \geq 0$, $f(x) = \cos(nx)$ is an eigenfunction of $T(f) = f'' - f$.

Solution:
We have $T(f) = -n^2 f - f = (-1 - n^2)f$. Also $\sin(nx)$ are eigenfunctions.

3) **T** F All symmetric real matrices are diagonalizable.

Solution:
This is a basic fact

4) T **F** There exists a 3×3 real symmetric matrix which is similar to $B = \begin{bmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

Solution:
A symmetric matrix has real eigenvalues. The given matrix B has two imaginary eigenvalues.

5) **T** F If A is any matrix, then both AA^T and $A^T A$ are orthogonally diagonalizable.

Solution:
Both matrices are square matrices. Both matrices are symmetric because $(AA^T)^T = (A^T)^T A^T = AA^T$ and $(A^T A)^T = A^T A$. Symmetric matrices are diagonalizable.

6) **T** F All orthogonal projections are diagonalizable

Solution:
This is a special case of the previous problem. Take an orthogonal basis in the space V, build the matrix A which has this basis as columns, then form AA^T. This is a symmetric matrix.

7) | T | F | (T highlighted)

If the regression line $y = ax + b$ obtained by fitting some data $\{(x_1, y_1), ..., (x_m, y_m)\}$ happens to contain all datapoints, then the corresponding least square solution of $A\vec{x} = \vec{b}$ is an actual solution of $A\vec{x} = \vec{b}$.

Solution:
This is almost tautological. The system of equations $A\vec{x} = \vec{b}$ is obtained by imagining all datapoints to be on the fitting curve. In general, this equation has no solution and we have to find the least square solution.

8) | T | F | (T highlighted)

$$\det(\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 1 & 2 & 2 \\ 2 & 2 & 1 & 2 \\ 2 & 2 & 2 & 1 \end{bmatrix}) = -7.$$

Solution:
The matrix $A + 1$ has entries 2 at every place and the eigenvalues 8 (with multiplicity 1) and 0 (with multiplicity 3). The matrix A has the eigenvalue 7 with multiplicity 1 and the eigenvalue -1 with multiplicity 3. The determinant of A is the product of the eigenvalues of A and is $(-1)^3 7 = -7$.

9) | T | F | (F highlighted)

There exists a symmetric 2×2 matrix A such that $A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ and $A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

Solution:
The two vectors are eigenvectors but these two vectors are not perpendicular.

10) | T | F | (T highlighted)

The kernel of the operator $T = (D - 2)^5$ is spanned by e^{2t}, te^{2t}, $t^2 e^{2t}$, $t^3 e^{2t}$, $t^4 e^{2t}$.

Solution:
The kernel is 5 dimensional and each of the functions indeed is in the kernel of T.

11) T F Let A be a 2×2 matrix. The system $\frac{dx}{dt} = Ax$ is asymptotically stable if and only if the eigenvalues of A have negative real parts.

Solution:
This is a theorem.

12) T F Let A be a 2×2 matrix. $\vec{0}$ is an asymptotically stable equilibrium of $\vec{x}(t+1) = A\vec{x}(t)$ if and only if all eigenvalues of A have negative real parts.

Solution:
For discrete dynamical systems, the norm of the eigenvalues matters, not the real part.

13) T F The subset of $X = C^\infty(\mathbb{R})$, the set of smooth functions of the real line, defined by $Y = \{f \in C^\infty(\mathbb{R}) : f(0) = 1\}$ is a linear subspace of X.

Solution:
It is not a linear space for example because Y does not contain the zero function $f(x) = 0$.

14) T F The subset of $C^\infty(\mathbb{R})$ defined by $Y = \{f \in C^\infty(\mathbb{R}) : f(0) = f''(2)\}$ is a linear subspace of X.

Solution:
If f, g are two functions in Y, then $f + g$ is a function in Y. If f is a function in Y, then λf is a function in Y. Also the zero function $f = 0$ is in Y.

15) T F The differential operator $T(f) = (D^2 + 8D + 17)f$ defined on $C^\infty(\mathbb{R})$ has as the image $C^\infty(\mathbb{R})$.

Solution:
We know that T is invertible.

16) T F If $\vec{0}$ is a least squares solution of $A\vec{x} = \vec{b}$, then $\vec{b} \in \text{im}(A)^\perp$.

Solution:
Yes, this is the main idea of the least square solution.

17) | T | **F** | If A is 2×2 matrix with $\det(A) < 0$, then the system $\frac{dx}{dt} = Ax$ has 0 as a stable equilibrium.

Solution:
One of the eigenvalues is negative.

18) | **T** | F | In the Fourier series expansion of the function $f(x) = x + 1$ on $[-\pi, \pi]$, the coefficients a_n belonging to $\cos(nx)$ are zero for all $n \geq 1$.

Solution:
The Fourier expansion of $f(x) = x + 1$ is the sum of the Fourier expansions of 1 and the odd function $g(x) = x$. The later has a pure sin expansion, the constant function 1 is its own Fourier expansion.

19) | T | **F** | $\vec{0}$ is a stable equilibrium of the discrete dynamical system $\vec{x}(t+1) = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} \vec{x} - 2\vec{x}$.

Solution:
The matrix $\begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}$ has trace -4. The sum of the eigenvalues is -4. For stability both eigenvalues would have to have absolute value < 1 which is not compatible with a trace -4.

20) | **T** | F | If A and B are diagonalizable matrices with the same eigenvectors, then $A + B$ is diagonalizable.

Solution:
The conjugating matrix S is the same for both matrices.

Problem 2) (10 points)

Match the following differential equations with the correct description. Every equation matches exactly one description. No justifications are necessary.

a)
$$\frac{d}{dt}x = 3x - 5y$$
$$\frac{d}{dt}y = 2x - 3y$$

b)
$$\frac{d}{dt}x = -4y + 2x^2 + 2x^3$$
$$\frac{d}{dt}y = 4y(1 - x^2)$$

c)
$$\frac{d}{dt}x = -x + 2y - y^2$$
$$\frac{d}{dt}y = 3x - y - xy - y^2$$

d)
$$\frac{d}{dt}x = 3x - 5y$$
$$\frac{d}{dt}y = x^2 + y^2 + 2$$

e)
$$\frac{d}{dt}x = 2y(x - y) - x$$
$$\frac{d}{dt}y = y(x - y) - y$$

Fill in 1),...,5) here.

a)	b)	c)	d)	e)

1) The equation has a stable equilibrium at $x = 1, y = 1$.
2) The equation has an unstable equilibrium at $x = 1, y = 1$.
3) The equation has a non-constant solution which stays on the line $x = y$.
4) The equation has a closed periodic trajectory.
5) The equation has no equilibria.

Solution:
1) matches with c). The Jacobian matrix at $(1, 1)$ has the eigenvalues $1, -4$.
2) matches with b). The Jacobian matrix at $(1, 1)$ has the trace 10 and can not be stable.
3) matches with e). If $x = y$, then $\frac{d}{dt}x = -x; \frac{d}{dt}y = -y$.
4) matches with a). The solution curves are on ellipses.
5) matches with d). Note that $\frac{d}{dt}y$ is never 0

a)	b)	c)	d)	e)
4	2	1	5	3

Problem 3) (10 points)

a) Find all 2×2 matrices which are both upper triangular and orthogonal.

b) Find a matrix A which has $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in the kernel of A.

c) Find a 2×2 matrix A such that $AA^T = A^T A$.

d) Find a matrix B such that $\det(BB^T) \neq \det(B^T B)$.

e) Find a 5×5 matrix A which has 0 as an eigenvalue with geometric multiplicity 3 and the eigenvalue 2 with geometric multiplicity 2.

f) Find a 2×2 matrix which is nondiagonalizable and which has $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ as an eigenvector.

Solution:

a) An orthogonal 2×2 matrix of determinant 1 is of the form $A = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$. If it is upper triangular, then $\sin(\alpha) = 0$ so that $\alpha = 0, \pi$. The only matrices are 1_2 and -1_2. An orthogonal 2×2 matrix of determinant -1 is of the form $A = \begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix}$. If it is upper triangular then $\sin(2\alpha) = 0$ so that $\alpha = 0, \pi$. The only matrices are $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. In total there are four matrices. You can also solve this by taking a general matrix A and forming $A^T A = 1_2$.
b) Take $A = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}$.
c) Any symmetric 2×2 matrix A satisfies $AA^T = A^T A$. For example, the identity matrix $A = 1_2$.
d) Take a non square matrix $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then $A^T A$ is a 1×1 matrix $A^T A = 1$ which has determinant 1 and AA^T is a projection matrix which has determinant.
e) Take a diagonal matrix $A = Diag(0, 0, 0, 2, 2)$.
f) Take a shear along the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Problem 4) (10 points)

Let $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

a) Find all possibly complex eigenvalues of A with their algebraic multiplicities.
b) Does A have a possibly complex eigenbasis? If so, find one.
c) Is A diagonalizable? Why or why not?
d) Let T be the linear transformation defined by $T(v) = Av$. Describe T geometrically.

Solution:
a) The characteristic polynomial is $(\lambda - 1)^2(\lambda + 1)^2$. The eigenvalues are $1, -1$ with algebraic multiplicities 2.

b) Yes, there is an eigenbasis: $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}. \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}. \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}.$

c) Yes, it is diagonalizable. We have a complete set of eigenvectors d) It permutes the standard basis vectors. It is a reflection in four dimensional space.

Problem 5) (10 points)

Find the function $f(x) = a + b\cos(x)$ which best fits the data

$$\begin{aligned} (x_1, y_1) &= (0, 1) \\ (x_2, y_2) &= (\pi/2, -1) \\ (x_3, y_3) &= (\pi, 1) \\ (x_4, y_4) &= (2\pi, 1) \end{aligned}$$

Solution:
Setting up the equations $a + b\cos(x_i) = y_i$ gives the system

$$\begin{aligned} a + b &= 1 \\ a &= -1 \\ a - b &= 1 \\ a + b &= 1 \end{aligned}$$

which can be written as $A\vec{x} = \vec{b}$, where $\vec{x} = [a, b]^T$ and $\vec{b} = [1, -1, 1, 1]^T$. The least square solution is given by $(A^TA)^{-1}A^T\vec{b}$. One has $A^TA = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}$ and $A^T\vec{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $(A^TA)^{-1}A^T\vec{b} = \begin{bmatrix} 5/11 \\ 2/11 \end{bmatrix}$. The best fit is given by the function $\boxed{f(t) = 5/11 + 2/11\cos(t)}$.

Problem 6) (10 points)

a) Find the solution of the differential equation $f'(t) + 3f(t) = e^{-2t}, f(0) = 0$.
b) Find the general solution of $f''(t) + 4f'(t) + 3f(t) = 1$.
with $f(0) = 1/3, f(1) = 1/3 + 1/e^3 - 1/e$.
c) Find the solution of $f''(t) = -4f(t)$ with $f(0) = 1, f'(0) = 2$.

Solution:
a) The general solution of the homogeneous equation $(D+3)f = 0$ is e^{-3t}. For a special solution, try $f = e^{-2t}$ which gives $(D+3)f = -2+3f = f$. The general solution is therefore $f(t) = e^{-2t} + c3^{-3t}$. For $t = 0$, this is $1 + c$ such that $c = -1$. $\boxed{\text{The solution is } f(t) = e^{-2t} - e^{-3t}.}$
b) The general solution of the homogeneous equation $(D^2+4D+3)f = (D+3)(D+1)f$ is $ae^{-3t} + be^{-t}$. A special solution of the inhomogeneous equation if $f(t) = 1/3$. Therefore, the general solution is $1/3 + ae^{-3t} + be^{-t}$. $\boxed{\text{The solution is } f(t) = 1/3 + e^{-3t} - e^{-t}.}$
c) The equation has the solutions $f(0)\cos(2t) + \frac{f'(0)}{2}\sin(2t) = \cos(2t) + \sin(2t)$.

Problem 7) (10 points)

a) (7 points) Find a 4×4 matrix A with entries $0, +1$ and -1 for which the determinant is maximal.
b) (3 points) Find the QR decomposition of A.

Solution:
a) This was a homework problem. The determinant is the volume of a parallelepiped. If its lengths are fixed, then it is maximal, if all sides are orthogonal. This can be achieved with $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$.
b) $A = Q \cdot R = (A/2) \cdot 2I_4$.

Problem 8) (10 points)

a) Diagonalize $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

b) Let A be the matrix $\begin{bmatrix} 7 & 4 & 0 \\ 4 & 0 & 4 \\ 0 & 4 & -7 \end{bmatrix}/9$, which has eigenvalues $-1, 0, 1$. Let P be the orthogonal projection onto the kernel of A. Find real numbers a, b, c such that $P = a(A - bI)(A - cI)$.

Hint: b) requires no calculations, rather thought).

Solution:
a) The eigenvalues are $1, 0, 2$. Note that the matrix is block diagonal and that the eigenvalues of $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ are the roots of $\lambda^2 - 2\lambda$ which are 0 and 2. The eigenvector to 1 is e_1. The vector $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ is in the kernel. The vector $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is the eigenvector to 2. These three vectors are orthogonal. Scaling leads to an orthonormal basis.
b) If $c = 1, b = -1$, then B annihilates the eigenvectors to $1, -1$. Let v be the eigenvector to 0. Then $Bv = a(-b)(-c)v$. In order that this is 1, we must have $a(-b)(-c) = 1$ or $a = -1$. The solution is $\boxed{a = -1, b = -1, c = 1}$.

Problem 9) (10 points)

Define $f = \sinh(x) = \frac{e^x - e^{-x}}{2}$ on $C^\infty([0, \pi])$ be a function on the interval $[0, \pi]$. Find a solution $u(x, t)$ of the heat equation $u_t = u_{xx}$ which satisfies $u(0, x) = f(x)$.

Hint. $\int \sinh(x) \sin(nx)\, dx = \frac{\cosh(x)\sin(nx) - n\cos(nx)\sinh(x)}{1+n^2}$. You can leave terms like $\sinh(\pi)$.

Solution:
$\sinh(x)$ is an odd function. So the Fourier coefficients are $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sinh(x)\sin(x)\, dx = \frac{2}{\pi}\int_0^\pi \sinh(x)\sin(nx)\, dx = \frac{2n\sinh(\pi)(-1)^{n+1}}{\pi(1+n^2)}$. The solution of the wave equation can now be written down:
$u(x, t) = \sum_{n=1}^\infty a_n e^{-n^2 t}\sin(nx)$ which is $\boxed{\sum_{n=1}^\infty \frac{2n\sinh(\pi)(-1)^{n+1}}{\pi(1+n^2)} e^{-n^2 t}\sin(nx)}$

Problem 10) (10 points)

a) Find the Fourier series of $|\sin(x/2)|$ on $C([-\pi, \pi])$.

b) Find $\sum_{n=1}^\infty (-1)^n \frac{1}{4n^2-1}$.

Solution:
a) The function is even so that it has a cos-Fourier series

$$f(x) = a_0/\sqrt{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \; .$$

We get

$$a_0 = \frac{2}{\pi} \int_0^{\pi} \sin(x/2) \frac{1}{\sqrt{2}} \, dx = \frac{2\sqrt{2}}{\pi}$$

and

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} \sin(x/2) \cos(nx) \; dx \\ &= \frac{1}{\pi} \int_0^{\pi} \sin(x/2 + nx) + \sin(x/2 - nx) \; dx \\ &= \frac{1}{\pi} [-\cos(x/2 + nx)/(1/2 + n) - \cos(x/2 - nx)/(1/2 - n)|_0^{\pi}] \\ &= \frac{2}{\pi} 0 - [-1/(1/2 + n) - 1/(1/2 - n)] \\ &= \frac{4}{\pi} \frac{1}{1 - 4n^2} \end{aligned}$$

$\boxed{a_n = \frac{4}{\pi} \frac{1}{1-4n^2}}$.
b) Evaluating f at $x = \pi$ gives $f(\pi) = 1$ and because $\cos(n\pi) = (-1)^n$ we have

$$1 = f(\pi) = \frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{1}{4n^2 - 1}$$

The solution is $(1 - 2/\pi)\pi/4$ which is equal to $\boxed{(2 - \pi)/4 = -0.285398.}$

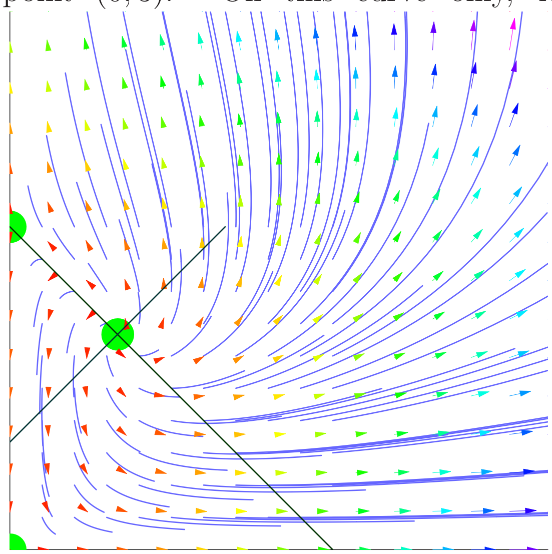
Problem 11) (10 points)

An ecological system consists of two species whose populations at time t are given by $x(t)$ and $y(t)$. The evolution of the system is described by the equation

$$\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x(x - y + 1) \\ y(x + y - 3) \end{bmatrix} \; .$$

a) Find all equilibrium points and nullclines of this system in $x \geq 0, y \geq 0$.
b) Sketch the vector field of this system in the first quadrant $x \geq 0, y \geq 0$ indicating the direction of the vector field along the nullclines and inside the regions determined by the nullclines.
c) Are there any stable equilibrium points? Justify your answers.
d) If both species start with positive populations, can either become extinct? Explain.

Solution:
a) The nullclines are $x = 0, y = 0, x - y + 1 = 0$ and $x + y - 3 = 0$. The equilibrium points are $(0, 0), (-1, 0), (0, 3)$ and $(1, 2)$ but only $(0, 0), (0, 3), (1, 2)$ are in the first upper right quadrant.
b) On the x-axes $y = 0$, the vector field is $(x^2 + x, 0)$ which is positive on the positive axes. On the y-axes, the field is $y^2 - 3y$ which is pointing up for $y > 3$ and pointing down for $y < 3$. The field spirals out at $(1, 2)$.
c) The Jacobian matrix is $\begin{bmatrix} 2x - y + 1 & -x \\ y & 2y - 3 + x \end{bmatrix}$. At $(1, 2)$ this is $\begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}$ which has eigenvalues $(3 \pm i\sqrt{7})/2$. $(0, 3)$, the Jacobian is $\begin{bmatrix} -2 & 0 \\ 3 & 3 \end{bmatrix}$. There is no asymptotic stability for any equilibrium point.
d) $\boxed{\text{Yes}}$, there is one curve which emerges from the equilibrium point $(1, 2)$ and which enters the equilibrum point $(0, 3)$. If we start on that curve but not on the equilibrium point $(1, 2)$, the solution with run into the equilibrium point $(0, 3)$. On this curve only, it is possible that the first species dies out.



Problem 12) (10 points)

Consider the linear differential equation

$$\begin{aligned} \frac{dx}{dt} &= ax + y \\ \frac{dy}{dt} &= ay \\ \frac{dz}{dt} &= -z \,. \end{aligned}$$

a) Write the system in the form $\frac{d}{dt}\vec{x} = A\vec{x}$, where A is a matrix.
b) For which parameters a is the system stable?

Solution:
a) The matrix is

$$A = \begin{bmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & -1 \end{bmatrix} .$$

b) In order that the system is stable, the eigenvalues of the matrix have to be all negative. This is equivalent to the fact that all the eigenvalues of $B = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$ have a negative real part. And this is true if $\boxed{a < 0}$.

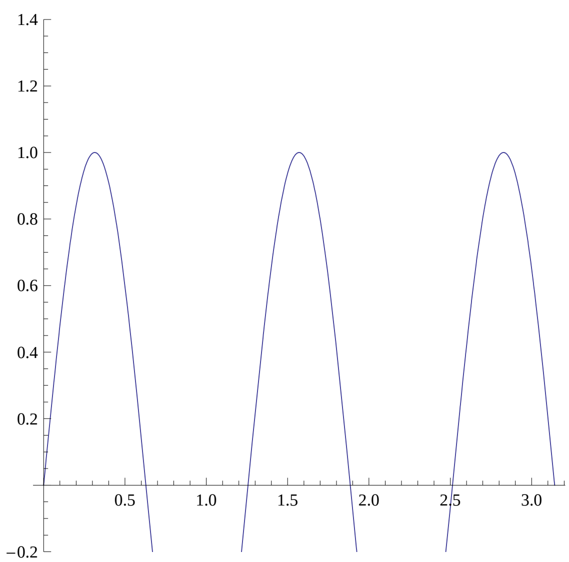
Problem 13) (10 points)

Find the solution of the modified wave equation $f_{tt} = 16 f_{xx} + f$ with initial string position

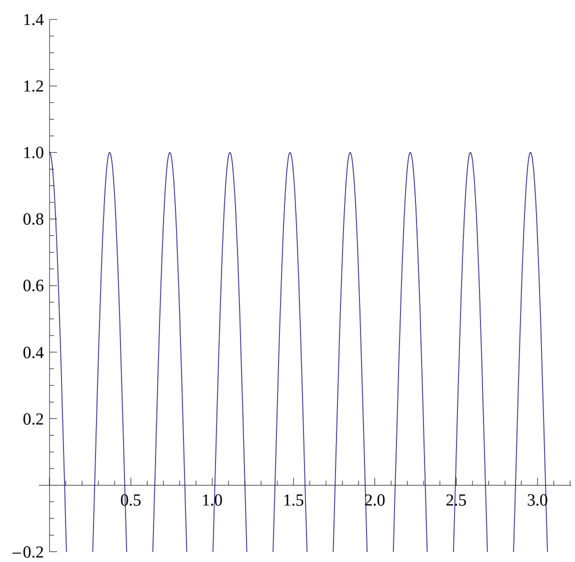
$$f(x, 0) = \sin(5x)$$

and initial string velocity

$$f_t(x, 0) = \sin(17x) .$$



Initial wave position on $[0, \pi]$.



Initial wave velocity on $[0, \pi]$.

Solution:
The Fourier decomposition of the initial wave position is $\sin(5x)$. The Fourier decompositon of the initial wave velocity is $\sin(17x)$ already. Note that the right hand side of the wave equation is of the form $16D^2 + 1$ which has eigenvalues $-c_n^2 = 1 - 16n^2$. The equation

$$u_{tt} = (1 - 16n^2)u = -c_n^2$$

has the solutions $\cos(\sqrt{16n^2 - 1}t)u(0) + \sin(\sqrt{16n^2 - 1}t)u'(0)/\sqrt{1 - 16n^2}$ because we know that the harmonic oscillator

$$u_{tt} = -c_n^2 u$$

has the solutions $\cos(c_n t)u(0) + \sin(c_n t)u'(0)/c$. The solution of the wave equation is

$$f(x,t) = \cos(\sqrt{16 \cdot 5^2 - 1}t)\sin(5x) + \sin(\sqrt{16 \cdot 17^2 - 1}t)\sin(17x)/\sqrt{16 \cdot 17^2 - 1} \ .$$

Problem 14) (10 points)

Because a higher wave is accelerated more, there is a force proportional to u and the partial differential equation is

$$u_{tt} = u_{xx} + 5u \ .$$

a) (3 points) Solve the system for initial condition $u(x,0) = \sin(3x) - \sin(7x)$ and $u_t(x,0) = 0$.
b) (3 points) Monster waves: Find the solution of the differential equation given in a), if the initial condition is $\sin(x)$.
c) (4 points) Verify that the wave height of the system

$$u_{tt} = cu_{xx} + bu$$

stays bounded at all times if the wind strength b is small enough or the string constant c is big enough. Verify that for $b > c$, we have solutions which explode as in b).

Solution:
a) We have $u_{tt} = T(u)$, where $T = D^2 + 5$ has eigenvalues $-n^2 + 5$ with eigenfunctions $\sin(nx)$. The functions $\sin(3x)$ and $\sin(7x)$ are eigenfunctions and we have

$$u(x,t) = \sin(3x)\cos(\sqrt{3^2 - 5}t) - \sin(7x)\cos(\sqrt{7^2 - 5}t) \ .$$

b) Now

$$u(x,t) = \sin(x)\cos(\sqrt{5 - 1}t) = \sin(x)\cos(2t)$$

explodes for $t \to \infty$.
c) We need $-cn^2 + b$ to be negative at all times so that all eigenfunctions $\sin(nx)$ move in a bounded way. For $b > c$, the eigenfunction $\sin(x)$ becomes unstable.