Name:

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MWF10 Ziliang Che
MWF10 Jeremy Hahn
MWF11 Rosalie Belanger-Rioux
MWF11 Yu-Wen Hsu
MWF12 Peter Garfield
TThu10 Oliver Knill
TThu11:30 Alex Perry
TThu11:30 Rong Zhou

- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1)	T	F	If u, v are two vectors in $\mathbf{R}^2$ which have length 1 and are perpendicular, then $u+v$ has length $\sqrt{2}$.
2)	T	F	The rank of a matrix is always larger or equal than the dimension of the kernel.
3)	T	F	The set of all 2×2 matrices A for which $A^2 = 0$ is a linear space.
4)	T	F	If A is a 1×1 matrix, then $\text{rref}(A)$ is either $[0]$ or $[1]$.
5)	T	F	If a system of linear equations has at least 2 solutions, then it has at least 4 solutions.
6)	T	F	If S is the matrix containing a basis $\mathcal{B}$ as columns, and A represents a transformation in the standard basis and $Av = Sv$ for every v, then in the basis $\mathcal{B}$, the transformation has same matrix $B = A$.
7)	T	F	The reflection A at the x axes is similar to the reflection B at the y axes. (Similar means that $B = S^{-1}AS$ for some S.)
8)	T	F	A 3×3 matrix can have $\dim(\text{im}(A)) = \dim(\ker(A))$.
9)	T	F	The rank of a 17×13 matrix can be 14.
10)	T	F	If $\{v_1, v_2, v_3, v_4\}$ is a set of linearly independent vectors spanning a linear subspace V of $\mathbf{R}^9$, then $\dim(V) = 4$.
11)	T	F	For a 10×6 matrix A, the dimension of $\ker(A)$ is at least 4.
12)	T	F	If A, B, and C are $n \times n$ matrices, then the property $A(BC) = (AB)C$ holds.
13)	T	F	Suppose A and B are $n \times n$ matrices. If A is invertible and B is not, then AB is not invertible.
14)	T	F	If a $n \times n$ matrix A has a nonzero vector v in the kernel, then it is not invertible.
15)	T	F	If $A\vec{x} = \vec{0}$ has two linearly independent solutions, where A is a 5×5 matrix, then $\text{rank}(A) \leq 2$.
16)	T	F	If A has the trivial kernel and $\vec{b}$ is in $\text{im}(A)$, then $A\vec{x} = \vec{b}$ has exactly one solution.
17)	T	F	If A and B are 2×2 matrices and AB is a rotation, then A and B are both invertible.
18)	T	F	If $\vec{v}$ is a nonzero vector in the kernel of A, then $\vec{v}$ is perpendicular to every row vector of A.
19)	T	F	For any two $n \times n$ matrices A and B, $(A+B)^2 = A^2 + 2AB + B^2$.
20)	T	F	There is a 2×3 matrix A and a 3×2 matrix B such that AB is the identity and $BA = 0$.

	Total

Problem 2) (10 points) No justifications are needed.

a) (5 points) Which of the following matrices can be put in row reduced echelon form in one single row reduction step (either swap, scale or subtract a row from an other row)?

Matrix	is in rref in one step	Matrix	is in rref in one step
$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$		$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	
$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$		$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$	
$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$		$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$	

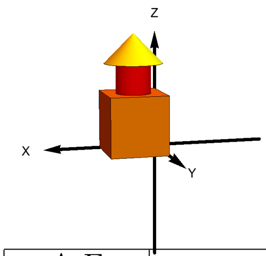
b) (3 points) Which sets are linear spaces

	The set of $\vec{x}$ which solve $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix} \cdot \vec{x} = 0$.
	The set of vectors $\vec{x}$ for which $x_1 + \cdots + x_n = 0$
	The image of the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}$.

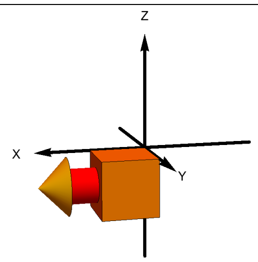
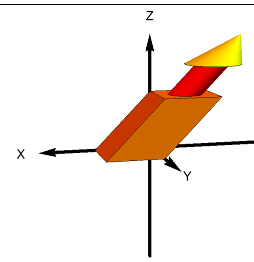
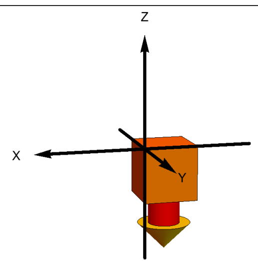
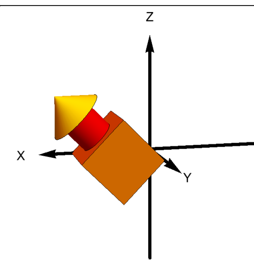
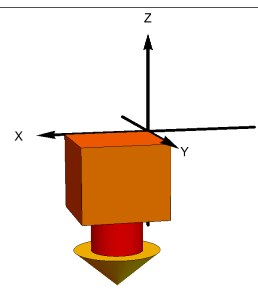
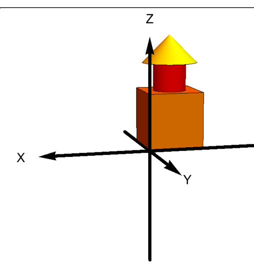
c) (2 points) One of the following matrices is not invertible, which one?

Matrix	not invertible
$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	
$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$	
$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}$	
$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	
$\begin{bmatrix} 5 & 4 & 3 & 2 \\ 4 & 3 & 2 & 0 \\ 3 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}$	

Problem 3) (10 points) No justifications are necessary.



a) (6 points) Match the matrices with the action of the transformation which maps a shape using a transformation given by matrices $A - F$.

A-F		A-F	
			
			
			

$$A=\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad B=\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad C=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$D=\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E=\begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \quad F=\begin{bmatrix} 1.5 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & -1.5 \end{bmatrix}$$

b) (4 points) Which of the following matrices either perform a rotation dilation or a reflection dilation, which a projection dilation or shear dilation. (A shear dilation is a composition of a shear and a dilation).

Matrix	rotation dilation	reflection dilation	projection dilation	shear dilation
$\begin{bmatrix} 5 & 0 \\ 6 & 5 \end{bmatrix}$				
$\begin{bmatrix} 5 & 6 \\ 6 & -5 \end{bmatrix}$				

Problem 4) (10 points)

Solve the following system of equations:

$$\begin{array}{rcl} x+y+z+w &=& 2 \\ x-y+z-w &=& 4 \end{array}$$

Your result should give a parametrization of all the solutions.

Problem 5) (10 points)

Find a basis of the kernel and image of the 15 puzzle matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 0 \end{bmatrix} .$$

You might have played the 15 puzzle as a kid. The puzzle was invented by Noyes Palmer Chapman a postmaster in Canastota, New York as early as 1874. Copies of an improved puzzle made their way to Syracuse, New York to Watch Hill and finally to Hartford from where it was sold in Boston, where it got produced on a larger scale starting in 1879. A version with permuted 15-14 tiles was sold too. It is unsolvable, driving many mad.

Problem 6) (10 points)

a) (5 points) Invert the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

by using row reduction on an augmented 3×6 matrix.

b) (5 points) Find a basis for the linear space of vectors perpendicular to

$$v = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} .$$

Problem 7) (10 points)

a) (5 points) Find the 3×3 matrix which reflects about the line through $v = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$.

b) (5 points) Find the 3×3 matrix which reflects about the plane $-x + 2z = 0$.

Problem 8) (10 points)

Given the following matrices

$$A = \begin{bmatrix} -1 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

compute:

a) (2 points) AB

b) (2 points) BA

c) (2 points) ACB

d) (2 points) BCA

e) (2 point) C^2

Problem 9) (10 points)

Motivated by $\pi = 3.141...$ and $e = 2.718...$ we look at the vectors $x = \begin{bmatrix} 3 \\ 1 \\ 4 \\ 1 \end{bmatrix}$ and $y = \begin{bmatrix} 2 \\ 7 \\ 1 \\ 8 \end{bmatrix}$.

a) (4 points) Find the normalized vectors $X_i = x_i - E[x], Y_i = y_i - E[y]$, where $E[x] = (x_1 + x_2 + x_3 + x_4)/4$ and the dot product $X \cdot Y$, as well as the lengths $|X|, |Y|$. This corresponds up to a normalization to the correlation and standard deviation of the random variables X, Y in statistics.

b) (6 points) Find the correlation coefficient

$$\frac{X \cdot Y}{|X||Y|}$$

which we know to be the cosine between the angle between X and Y.