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- Start by writing your name in the above box and check your section in the box to the left.

- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.

- Do not detach pages from this exam packet or un-staple the packet.

- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.

- No notes, books, calculators, computers, or other electronic aids can be allowed.

- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (20 points) No justifications are needed.

1) T **F**   The set of smooth functions in $C^\infty(R)$ with $f(2) = 3$ is a linear space.

Solution:
It does not contain the zero function.

2) **T** F   If matrix A is invertible, then rref(A) must be invertible too.

Solution:
If A is invertible, then rref(A) is the identity matrix and also invertible.

3) T **F**   If A is an invertible matrix, and $B = $ rref(A). Then $A^{-1} = B^{-1}$.

Solution:
If A is invertible, then $B = $ rref(A) is the identity matrix and so also B^{-1} is the identity matrix. But A^{-1} is not necessarily the identity matrix.

4) T **F**   There is a linear subspace of $\mathbf{R}^7$ that contains exactly seven vectors.

Solution:
A linear subspace contains either exactly one vector (the zero vector), or it contains infinitely many vectors.

5) T **F**   There exists a 7×3 matrix that has rank 7.

Solution:
The rank of a $n \times m$ matrix can not be larger than n nor larger than m. In this case, it can not be larger than 3.

6) T **F**   The circle $x^2 + y^2 = 1$ is the kernel of a linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$.

Solution:
The circle can not be the kernel. It is not a linear space.

7) **T** F If a matrix A is similar to a matrix B and A is invertible, then B is invertible.

Solution:
$B = S^{-1}AS$, then $B^{-1} = S^{-1}A^{-1}S$.

8) T **F** A reflection about the line $x + y = 1$ is a linear transformation.

Solution:
The does not leave the point $\vec{0} = (0, 0)$ invariant.

9) **T** F If $A^2BA^3 = I_3$ for 3×3 matrices A, B, then B is invertible.

Solution:
If B were not invertible, the image of B would be 0 or 1 or 2 dimensional. But that means that the image of BA^3 is 0 or 1 or 2 dimensional. By the dimension formula, the kernel of BA^3 is 3 or 3 or 2 or 1 dimensional meaning that there is a vector v with $BA^3v = 0$. But then also $A^2BA^3v = 0$ and v is in the kernel of A^2BA^3, which is not possible because I_3 has only a trivial kernel.

10) **T** F For any reflection A about the origin in R^2, there exists a 2×2 matrix B such that $A = B^2$.

Solution:
Take a rotation about 90 degrees. Then the square B^2 is a rotation by 180 degrees which is a reflection at the origin.

11) T **F** For any 2×2 matrix, we always have rank(A) = rank(A^2).

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Solution:
The rank of A^2 can be smaller. An example is $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

12) T **F** It is possible that a system $Ax = b$ has a unique solution for some b if A is a 2×3 matrix.

Solution:
No, there are either 0 or infinitely many solutions.

13) T **F** A reflection about the x-axis is similar to a rotation by 90 degrees in the plane.

Solution:
A reflection matrix A satisfies $A^2 = I_2$ but a rotation satisfies $B^2 = -I_2$. They can not be similar.

14) T **F** There is a 6×4 matrix for which the kernel has dimension 5.

Solution:
There can be maximally 4 variables because the matrix has only 4 columns.

15) T **F** For any two 2×2 matrices A, B, the identity $(A - B)(A^2 + AB + B^2) = A^3 - B^3$ holds.

Solution:
This is wrong if $AB \neq BA$.

16) **T** F The column vectors of a 2×2 rotation-dilation matrix form an orthogonal basis.

Solution:
Yes, as they are the images of the basis vectors and the angles do not change

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17) **[T]** [F] If A is a non-invertible square matrix then $\mathrm{rref}(A)$ has at least one row of zeros.

Solution:

18) [T] **[F]** The plane $x + y + z = 1$ in space is the image of a linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^3$.

Solution:
The image of a linear transformation is a linear space. Especially, it has to contain the origin. The plane under consideration does not contain 0.

19) [T] **[F]** For any $n \times n$ matrix A, the identity $\ker(A^3) = \ker(A^2)$ holds.

Solution:
The kernel can change. Take a matrix for which A^3 is zero and A^2 nonzero.

20) **[T]** [F] If A is a 3×5 matrix of rank 3, then $A\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ has infinitely many solutions $\vec{x}$.

Solution:
There is a solution because the image is 3 dimensional. There is a two dimensional kernel. Because the image is the entire space R^3, we have at least one solution.

Total

a) (5 points) Match the following transformations with their names. Choices can appear multiple times. A shear dilation is a shear composed with a scaling, a rotation dilation is a rotation composed with a scaling, a reflection dilation is a reflection composed with a scaling, a projection dilation is a projection composed with a scaling. The scaling factors can also be 1 of course.

Matrix	Enter A-D here.
a) $\begin{bmatrix} 4 & 3 \\ 3 & -4 \end{bmatrix}$	
b) $\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$	
c) $\begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}$	
d) $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$	
e) $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$	
f) $\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$	
g) $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$	
h) $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$	

A) Reflection dilation

B) Shear dilation

C) Projection dilation

D) Rotation dilation

Solution:
a) A
b) C
c) D
d) B
e) B
f) D
g) C
h) A

b) (5 points) Match the matrices with their actions:

A-J	domain	codomain		A-J	domain	codomain

$$A \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad\qquad F \quad \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$B \quad \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \qquad\qquad G \quad \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$C \quad \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \qquad\qquad H \quad \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$D \quad \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad\qquad I \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$E \quad \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \qquad\qquad J \quad \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Solution:

I, B

E, G

J, A

F, H

D, C

Problem 3) (10 points) No justifications are needed.

a) (5 points) Check the boxes which apply.

matrix	similar to $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	invertible
$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$		
$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$		
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$		
$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$		
$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$		

Solution:

Only the last two are similar to the reflection. All except the first are invertible.

b) (5 points) Which of the following sets are linear spaces, which are linear transformations, which of them are none?

| object	linear space	linear transformation
$T(x, y) = x + y$		
$\{(x, y) \mid x + y = 1\}$		
The identity matrix		
$B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$		
$\{(x, y) \mid x = y\}$		
The image the zero matrix		

Solution:
linear trafo
nothing
linear trafo
linear trafo
linear space
linear space

Problem 4) (10 points)

Consider the system of linear equations

$$
\begin{vmatrix}
x & & & + & u & & & & & = & 3 \\
& y & & & & + & v & & & = & 5 \\
& & z & & & & & + & w & = & 9 \\
x & + & y & + & z & & & & & = & 8 \\
& & & u & + & v & + & w & = & 9
\end{vmatrix}
$$

Do we have infinitely many solutions, zero solutions or exactly one solution? If there are solutions, find all of them.
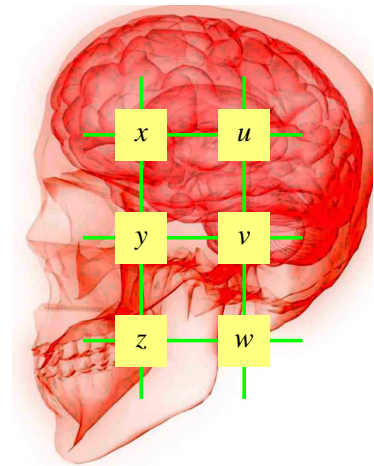
The problem appears in **tomography** like magnetic resonance imaging. A scanner can measure averages of tissue densities along lines. The task is to compute the actual densities.

Solution:

The augmented matrix is

$$
\left[
\begin{array}{cccccc|c}
1 & 0 & 0 & 1 & 0 & 0 & 3 \\
0 & 1 & 0 & 0 & 1 & 0 & 5 \\
0 & 0 & 1 & 0 & 0 & 1 & 9 \\
1 & 1 & 1 & 0 & 0 & 0 & 8 \\
0 & 0 & 0 & 1 & 1 & 1 & 9
\end{array}
\right].
$$

Remove the sum of the first three rows from the 4th, then change sign of the 4'th row:

$$
\left[
\begin{array}{cccccc|c}
1 & 0 & 0 & 1 & 0 & 0 & 3 \\
0 & 1 & 0 & 0 & 1 & 0 & 5 \\
0 & 0 & 1 & 0 & 0 & 1 & 9 \\
0 & 0 & 0 & 1 & 1 & 1 & 9 \\
0 & 0 & 0 & 1 & 1 & 1 & 9
\end{array}
\right].
$$

Now subtract the 4th row from the last to get a row of zeros, then subtract the 4th row from the first. This is already the row reduced echelon form.

$$
\left[
\begin{array}{cccccc|c}
1 & 0 & 0 & 0 & -1 & -1 & -6 \\
0 & 1 & 0 & 0 & 1 & 0 & 5 \\
0 & 0 & 1 & 0 & 0 & 1 & 9 \\
0 & 0 & 0 & 1 & 1 & 1 & 9 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\right].
$$

The first 4 columns have leading 1. The other 2 variables are free variables r, s. We write the row reduced echelon form again as a system and get so the solution:

$$
\begin{aligned}
x &= -6 + r + s \\
y &= 5 - r \\
z &= 9 - s \\
u &= 9 - r - s \\
v &= r \\
w &= s
\end{aligned}
$$

There are infinitely many solutions. They are parametrized by 2 free variables.

Problem 5) (10 points)

Let A be the matrix of a reflection dilation $T(x) = T_2(T_1(x))$, where the reflection T_1 is done

at the line $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and where the scaling is $T_2(x) = 2x$.

a) (5 points) Find a suitable basis for this problem in which the transformation is given by a diagonal matrix B.

b) (5 points) Find the matrix A of the transformation T in the standard basis.

Solution:

a) Take as the first basis vector is the vector in the axis rotation and take two other vectors perpendicular to it. For example

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}.$$

This leads to the S-matrix

$$S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}.$$

In that basis, the transformation is

$$B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

b) The inverse of S is

$$S^{-1} = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & -1/2 \end{bmatrix}.$$

We get

$$A = SBS^{-1} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & -2 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$$

Problem 6) (10 points)

Find a basis of the image and kernel of the following matrix and state what the rank-nullety theorem tells in this situation.

$$A = \begin{bmatrix} 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ 2 & 1 & 0 & 0 & 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 0 & 0 & 1 & 2 \\ 2 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{bmatrix}.$$

Solution:

Row reducing A (we wanted to see the work) gives

$$\text{rref}(A) \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

The first three columns have leading 1's. The first two columns of the matrix A form therefore the image of A:

$$\mathcal{B}_{im}(A) = \left\{ \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} \right\} .$$

We have 5 free variables and get from the row reduced echelon form a basis of the kernel

$$\mathcal{B}_{ker}(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \right\} .$$

The rank-nullety theorem is confirmed because $\dim(im(A)) + \dim(ker(A)) = 3 + 5 = 8$ is the number of columns.

Problem 7) (10 points)

Find a matrix A such that the image of the matrix $B = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}$ coincides with the kernel of A.

Solution:
The two column vectors of B are linearly independent so that the image of B is a two dimensional space. We need a matrix A for which

$$A \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \vec{0}, A \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \vec{0} \ .$$

Every row of A must be perpendicular to the two columns of B. In other words, every column x of the matrix A satisfies

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 3 & 2 & 1 \end{bmatrix} \vec{x} = \vec{0}$$

so that the column vectors of A can be computed as the kernel of the matrix to the left. To compute the kernel, we row reduce this 2×4 matrix and get

$$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \vec{x} = \vec{0}$$

which can be written as $x = r - s, y = -r; z = r, w = s$ The kernel of this matrix A^T is spanned by

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

We can take these vectors as row vectors of A to get

$$A = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 \end{bmatrix} \ .$$

Also the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

would work.

Problem 8) (10 points)

Describe the transformation $T(x) = Ax$ with

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

in the basis $\mathcal{B}$ given by the column vectors of the matrix

$$S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \ .$$

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In other words, find the 3×3 matrix B which describes T in the $\mathcal{B}$ coordinates.
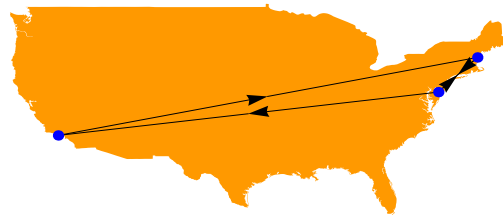
Solution:

We get $B = S^{-1}AS = \begin{bmatrix} 0 & 2 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$."

Problem 9) (10 points)

An airline services Boston, New York, Los Angeles. It flies from New York to Los Angeles, from Los Angeles to Boston and from Boston to New York as well as from New York to Boston.

The connection matrix is

$$\begin{bmatrix} & BO & NY & LA \\ BO & 0 & 1 & 0 \\ NY & 1 & 0 & 1 \\ LA & 1 & 0 & 0 \end{bmatrix}.$$

a) (7 points) To find the number of different round trips of length 8 starting from Boston, one can compute $A^8 = A \cdot A \cdot A \cdot A \cdot A \cdot A \cdot A \cdot A$ with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

and to look up the first entry in the first row of A^8. Compute the matrix A^8 and find so the number of round trips of length 8.

b) (3 points) Find the 2×2 matrix B such that

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} B \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Hint to a). Compute first $U = A \cdot A$, then $V = U \cdot U$ and finally $A^8 = V \cdot V$.

Solution:

a) By the way, this is a useful application of linear algebra to combinatorics. If we wanted to find out the number of round trips of length 100, we could proceed in the same way. Here:

$$A^8 = \begin{bmatrix} 4 & 3 & 2 \\ 5 & 4 & 3 \\ 3 & 2 & 2 \end{bmatrix} .$$

The number of round trips of length 8 are 4.

b)

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} .$$

Problem 10) (10 points)

A **two person zero-sum game** for two players Ana and Bob (Mathematicians like palindromes) is described by the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 5 \end{bmatrix}$$

The interpretation is that player "Ana" has two possible moves and player "Bob" has 3 possible moves. If Ana for example makes the first move and Bob makes the 2nd move, then the payoff for Bob to Ana is $A_{12} = 2$. If the row vector $a = [a_1 a_2]$ encodes the probabilities that Ana makes the moves and the column matrix $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ encodes the probability of the moves of Bob, then the **expected payoff** is

$$E(a, b) = aAb .$$

a) (4 points) Assume Ana choses her moves with equal probability $1/2, 1/2$ and Bob choses his decisions with equal probability $1/3, 1/3, 1/3$. What is the expected payoff?

b) (3 points) Assume Bob fixes his strategy and keeps his equal probability move strategy from a). What strategy a maximizes the expected payoff for Ana?

c) (3 points) Assume now Ana fixes her fifty/fifty strategy. What strategy b minimizes the expected payoff Bob has to pay to Ana?

Solution:
This problem is just a problem to matrix multiplication. It introduces some terminology from **Game theory** which would lead to much more interesting problems.

a) $a.A.b = 11/6$

b) $A.b = [2, 5/3]$. The best strategy is $a = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

c) $a.A = [1, 1/2, 4]$. The least loss for Bob is achieved, if he choses the strategy $b = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

P.S. the interesting questions start if both players do not know the strategy of the opponent. What is the best bet? This question is at the heart of **game theory**. It looks for example for **Nash equilibria**, strategies a and b for Ana and Bob such that a is the best strategy for b and b is the best strategy for a. Find the Nash equilibria can be a computationally difficult task if the pay off matrices are large.