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- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (20 points) No justifications needed

1) T F (F marked) The plane $x + y + z - 1 = 0$ is the kernel of a linear transformation T.

Solution:
The kernel of a transformation is a linear space. Especially, it has to contain the origin. The plane under consideration does not contain 0.

2) T F (F marked) For any matrix A the identity $\text{im}(A^2) = \text{im}(\text{rref}(A^2))$ holds.

Solution:
The image does change under row reduction. Already switching two rows changes the image in general. This does not change when considering A^2. Take a projection $A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ onto the line $x = y$ for example. It satisfies $A^2 = A$. After row reduction, $\text{rref}(A)$ is projection onto the x axes. The image has changed.

3) T F (T marked) For any matrix A, one has $\ker(A) = \ker(\text{rref}(A))$.

Solution:
The kernel of a matrix does not change under row reduction because the kernel are all vectors which are perpendicular to all row vectors of A and linear combinations of row vectors keep staying perpendicular to x.

4) T F (F marked) There is a 4×8 matrix whose kernel is 3-dimensional.

Solution:
The image is maximally 4 dimensional because there can be maximally 4 leading 1 in $\text{rref}(A)$. By the dimension formula, the kernel is at least 4 dimensional and so 4 free variables.

5) T F (T marked) There is a 2×2 matrix for which $A^2 = -I_2$.

Solution:
Look for rotations. $-I_2$ is a rotation by 180 degrees. So, a rotation by 90 degrees will do. This is by the way a matrix representation of the complex number i. If you ever wondered whether i, the square root of 1 exists, here it is.

6) | T | **F** | If three vectors v_1, v_2, and v_3 are in a plane, they are independent.

Solution:
They are dependent. If the three vectors span the plane, one can express v_3 as a linear combination of v_1 and v_2. If the three vectors are in a line, then one can express one as a multple of an other. In any case, they are dependent.

7) | T | **F** | If S and A are invertible $n \times n$ matrices, then $(SAS^{-1})^{-1} = S^{-1}A^{-1}S$.

Solution:
The correct answer is $SA^{-1}S^{-1}$. To derive this, use the fomula $(UV)^{-1} = V^{-1}U^{-1}$ twice.

8) | T | **F** | If all entries of a 2×2 matrix A are nonzero, then the inverse of A exists.

Solution:
A counterexample is the matrix which has all entries equal to 1. This matrix has rank 1. A matrix, in which all entries are equal to a constant has rank 1. It can only be invertible for nxn matrices with $n = 1$.

9) | **T** | F | For any square matrix A, the image of A^7 is contained in the image of A.

Solution:
If $y = A^7x$, then $y = A(A^6x)$ so that y is also in the image of A. It is the image of $z = A^6x$.

10) | T | **F** | If A is a matrix, let B be the matrix for which the order of the columns are reversed. Then $\text{rref}(A) = \text{rref}(B)$.

Solution:
Take $1x4$ matrices $A = [1, 2, 3, 1]$ and $B = [1, 3, 2, 1]$. We have rref$(A) = A$ but rref$(B) = B$ differ.

11) T F If the columns of a $n \times n$ matrix form a basis in $\mathbb{R}^n$, then the rows also form a basis in $\mathbb{R}^n$.

Solution:
If the columns form a basis, then A is invertible and then the rows form a basis too. If the rows would not form a basis, they would be linearly dependent and row reduction would give you a zero row.

12) T F If $B^2 = A$, then B is called the square root of A. Every 2×2 matrix A has either 0 or 1 or 2 square roots.

Solution:
Already the identity matrix has at least 4 square roots: $\text{diag}(1, 1), \text{diag}(1, -1), \text{diag}(-1, 1), \text{diag}(-1, -1)$. There are many more. The identity matrix for example has every reflection as a square root.

13) T F If A is an invertible 2×2 matrix and $\mathcal{B}$ is the basis of the column vectors, then $[A]_{\mathcal{B}}$ is diagonal.

Solution:
This would only be true, if Av is a multiple of v.

14) T F There exists a linear transformation whose image consists of exactly 6 distinct points.

Solution:
If $v \neq 0$ is in the image, then λv is in the image for every λ. This means that if the image contains a point different from 0, it contains infinitely many points.

15) T F $\mathcal{B} = \{2e_1, 2e_2\}$ is a basis of $\mathbb{R}^2$ for which $[v]_{\mathcal{B}} = v$ for any v.

Solution:
The coordinate transformation S is 2 times the identity. Use the formula $S^{-1}v = v/2$.

16) | T | F |

The dimension of the image of a matrix A is equal to the dimension of the image of the matrix $\mathrm{rref}(A)$.

Solution:
While image changes under row reduction, the dimension of the image does not change.

17) | T | F |

There exists an invertible $n \times n$ matrix whose inverse has rank $n - 1$.

Solution:
The inverse is invertible too and has therefore rank n also.

18) | T | F |

If A is the reflection about a line L and B is the reflection about the plane $V = L^{\perp}$ perpendicular to L, then $A = -B$.

Solution:
Choose a good basis for which $A = diag(-1, -1, 1)$ and $B = diag(1, 1, -1)$. Now $A + B = 0$.

19) | T | F |

The set of vectors in space satisfying $x + y + z = 1$ form a linear space of dimension 2.

Solution:
The dimension is right but its not a linear space.

20) | T | F |

If A, B are given $n \times n$ matrices, then there is a unique $n \times n$ matrix X satisfying $(A + X)B = A$ if B is invertible.

Solution:
Indeed, we can solve for $X = (A - AB)B^{-1}$.

| | Total |

Problem 2) (10 points)

Match the matrices to their rref, enter R, U, V, W in the right order below:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad V = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad W = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

rref $(A) =$	rref $(B) =$	rref $(C) =$	rref $(D) =$

b) Find the projection matrix onto the space spanned by the vectors $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}/2$, $\vec{w} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}/2$.

Solution:

$\text{rref}\,A = \begin{bmatrix} 1,0,0 \\ 0,1,0 \\ 0,0,1 \end{bmatrix} = U.$

$\text{rref}\,B = \begin{bmatrix} 1,0,1 \\ 0,1,0 \\ 0,0,0 \end{bmatrix} = W.$

$\text{rref}\,C = \begin{bmatrix} 1,1,1 \\ 0,0,0 \\ 0,0,0 \end{bmatrix} = V.$

$\text{rref}\,D = \begin{bmatrix} 1,1,0 \\ 0,0,1 \\ 0,0,0 \end{bmatrix} = R.$

rref (A) =	rref (B) =	rref (C) =	rref (D) =
U	W	V	R

b) Form $Q = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix}$ then compute $QQ^T = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$.

Problem 3) (10 points)

Each of the following matrices matches with a transformation below:

$$A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ -\sqrt{2} & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$D = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \quad E = \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix} \quad F = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$$

rotation	dilation	rotation dilation	shear	projection	reflection

Each of the following spaces R, U, V, W below is equal to one of the spaces K, L, M, N. No justification is necessary.

$$R = \text{im}\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \quad U = \text{im}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \quad V = \text{im}\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad W = \text{im}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$K = \ker \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \quad L = \ker \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$M = \ker \begin{bmatrix} 0 & 1 & -1 & 0 \\ 2 & -1 & -1 & 2 \end{bmatrix} \quad N = \ker \begin{bmatrix} 0 & 2 & -2 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix}$$

R =	U =	V =	W =

Solution:

rotation	dilation	rotation dilation	shear	projection	reflection
A	F	E	B	C	D

R =	U =	V =	W =
K	K	L	M or N

Problem 4) (10 points)

Solve the following system of linear equation by doing row reduction of the augmented matrix. In each step you have to use one of three basic row reduction steps. Write in each step, what you do:

$$\begin{array}{ccccccc} x & & +z & & = & 1 \\ & y & +z & +q & = & 2 \\ x & & +z & +q & = & 2 \\ x & +y & & +q & = & 1 \end{array}$$

Solution:
We do row reduction to the augmented matrix. Here are the main steps:

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 & 1 \end{array}\right] \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 & 0 \end{array}\right] \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 0 & -2 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

The solution is $x = 0, y = 0, z = 1, q = 1$.

Problem 5) (10 points)

Two matrices A, B are called **similar** if there exists an invertible matrix S such that $B = S^{-1}AS$.

a) (2 points) Is it possible that the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ be similar to a reflection about a line? Show your reasoning.

b) (3 points) What is the matrix A in the basis $\mathcal{B} = \{v_1, v_2, v_3\} = \{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \}$?

c) (3 points) Find the inverse of the matrix S which contains the above basis as column vectors v_1, v_2, v_3. Use row reduction to find the inverse.

d) (2 points) Express $\vec{v} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$ as a linear combination of v_1, v_2 and v_3.

Solution:
a) No. If two matrices A and B are similar and one of them is not invertible, then the other is not invertible. A reflection is invertible, the matrix A is not invertible because it contains two equal columns. An other solution is to look at A^2. It is not the identity. A reflection B has the property that $B^2 = I_3$. If $A \sim B$, then $A^2 \sim B^2 = I_3$ but A^2 is not the identity.

b) $B = S^{-1}AS = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$.

c) $S^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

d) $S^{-1} \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} = 3v_3$.

Problem 6) (10 points)

Here e_1, e_2, e_3 denote the standard basis vectors in $\mathbb{R}^n$.

a) (3 points) Find the 3×3 matrix A which is the reflection about the xz-plane.

b) (2 points) Find the 3×3 matrix B which maps e_1 to e_2, e_2 to e_3 and e_3 to e_1.

c) (2 points) Find the 3×3 matrix C which scales every vector by a factor 2.

d) (3 points) What is the transformation CBA which first reflects, then rotates and finally scales.

Solution:
The key was to look at the images of the basis vectors and put them as column vectors.

a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

b) $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

d) Either do the matrix multiplication explicitely or look what happens to the basis vectors when applying the three transformations: the answer is: $\begin{bmatrix} 0 & 0 & 2 \\ 2 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix}$.

Problem 7) (10 points)

Find a basis for the image and the kernel of the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 2 & 3 & 4 & 3 & 2 \\ 3 & 4 & 5 & 4 & 3 \\ 4 & 5 & 6 & 5 & 4 \\ 5 & 6 & 7 & 6 & 5 \end{bmatrix} .$$

Solution:
To row reduce A, note the structure of the rows. It is best to subtract the previous row from the next row first

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

The first two rows are pivot columns. The last three columns are redundant columns. A basis for the image is

$$\mathcal{B} = \{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} , \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} \} .$$

To get a basis for the kernel, write the system down

$$\begin{aligned} x &= z - w \\ y &= -2z - v \\ z &= z \\ v &= v \\ w &= w \end{aligned}$$

so that

$$\mathcal{B} = \{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} , \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} , \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \} .$$

is a basis for the image.

Problem 8) (10 points)

We are given the rotation dilation matrix $A = \begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$.

a) (3 points) Find a matrix B such that $B^2 = A$.

b) (4 points) Write down the matrix B^{17}. We need a numerical result which can involve powers of numbers.

c) (3 points) Can A be the product of a projection at a line L and a reflection Q at a second line K?

Solution:
a) The transformation A is a rotation dilation. The dilation factor is $\sqrt{1+3} = 2$. The angle is $\pi/3$. The transformation B is a rotation dilation by a scaling factor $\sqrt{2}$ and angle $\pi/6 = 30°$ which is

$$\sqrt{2}\begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}.$$

After 17 iterations, the dilation factor is $2^{17/2}$ and the angle is $17\pi/6 = 18\pi/6 - \pi/6 = -5\pi/3$. The matrix is

$$2^{17/2}\begin{bmatrix} \cos(5\pi/6) & -\sin(5\pi/6) \\ \sin(5\pi/6) & \cos(5\pi/6) \end{bmatrix} = 2^{17/2}\begin{bmatrix} -\sqrt{3}/2 & -1/2 \\ 1/2 & -\sqrt{3}/2 \end{bmatrix}.$$

c) No. A product B of a projection and reflection is not invertible. It has rank 1. But the transformation A is invertible. If $B = S^{-1}AS$, then either both A and B are invertible, or both are not invertible.

Problem 9) (10 points)

Let A be a 3×3 matrix such that $A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

a) (3 points) Why is $\text{im}\,(A^2)$ a subspace of $\ker\,(A)$?

b) (4 points) Find all possible values for $\text{rank}(A)$.

c) (3 points) Give an example of such a matrix A for each possible rank.

Solution:
a) If $x \in \text{im}(A^2)$, then then $x = A^2y$ and $Ax = A^3x = 0$ so that x is in the kernel.
b) It can not be 3 because otherwise, A and A^3 would be invertible. The rank can be $0, 1, 2$ although and c) shows that all of them can be achieved:
c) $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Problem 10) (10 points)

Let A be a 3×3 matrix with $A^3 = 0$, the zero matrix.

a) (3 points) Compare $\ker\,(A)$ and $\text{im}\,(A)$ and $\text{im}\,(A^2)$. How are they related?

b) (2 points) Is A invertible?
c) (2 points) Is $A+I_3$ invertible where I_3 is the identity matrix? Hint. Look at $B = I_3 - A + A^2$.
d) (3 points) Give examples of matrices A exhibiting all the relations you found in part a) and c).

Solution:
a) The image of A^2 is in the kernel of A. The image of A contains the image of A^2.
b) If A were invertible, A^3 would be. This contradiction shows that A can not be invertible.
c) Yes, it is aways inverible. The inverse is the matrix $B = I_3 - A + A^2$ given in the hint.
d) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.