

Name:	
MWF 9 Fabian Haiden	• Start by writing your name in the above box and check your section in the box to the left. • Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it. • Do not detach pages from this exam packet or unstaple the packet. • Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit. • No notes, books, calculators, computers, or other electronic aids can be allowed. • You have 90 minutes time to complete your work.
MWF 10 Ziliang Che	
MWF 10 Jeremy Hahn	
MWF 11 Rosalie Belanger-Rioux	
MWF 11 Yu-Wen Hsu	
MWF 12 Peter Garfield	
TThu 10 Oliver Knill	
TThu 11:30 Alex Perry	
TThu 11:30 Rong Zhou	

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

- 1) ☐ T ☐ F If T is an invertible linear transformation, then its inverse is a linear transformation too.
- 2) ☐ T ☐ F If T is a non-invertible transformation from $R^2 \rightarrow R^2$ then T can not be a linear transformation.
- 3) ☐ T ☐ F The matrix $A = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$ is an example of a rotation dilation matrix.
- 4) ☐ T ☐ F If a 3×3 matrix A is invertible, then its rank must be 3.
- 5) ☐ T ☐ F There is a 11×11 matrix A for which the rank of A is equal to the nullity of A .
- 6) ☐ T ☐ F There is a 4×3 matrix with a 3-dimensional kernel.
- 7) ☐ T ☐ F There is a 4×3 matrix with 4-dimensional image.
- 8) ☐ T ☐ F If a system of equations $Ax = b$ with nonzero b has at least one solution, then the space of solutions is a linear space.
- 9) ☐ T ☐ F If a system of equations has only $\vec{0}$ as a solution, then it is called inconsistent.
- 10) ☐ T ☐ F The map $T(x) = x + 1$ on the real line R^1 is an example of a linear transformation.
- 11) ☐ T ☐ F Since the map $T(x) = 1 - x$ satisfies $T(T(x)) = x$, it is a reflection and so a linear map.
- 12) ☐ T ☐ F There is a 31×32 matrix which has a trivial kernel.
- 13) ☐ T ☐ F The kernel of the 1×3 matrix $[1 \ 1 \ 1]$ is the plane $x + y + z = 0$.
- 14) ☐ T ☐ F If A is a 2×2 matrix satisfying $A^2 = 0$, then $(I_2 + A)$ is the inverse of $(I_2 - A)$.
- 15) ☐ T ☐ F If a 2×2 matrix satisfies the relation $A^2 = A$ then $A = 0$ or $A = I_2$.
- 16) ☐ T ☐ F The subset of functions f in C^∞ satisfying $f(4) = f'(4) = 0$ is a linear space.
- 17) ☐ T ☐ F If A, B, S are invertible 2×2 matrices satisfying $A = SBS^{-1}$ then $A^2 = SB^2S^{-1}$.
- 18) ☐ T ☐ F The set $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ is a basis for the kernel of $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.
- 19) ☐ T ☐ F There exist two 2×2 matrices A, B of rank 1 for which AB has rank 2.
- 20) ☐ T ☐ F Every basis of $\mathbf{R}^7$ contains exactly 7 vectors in it.

Total

Problem 2) (10 points) No justifications are needed.

a) (3 points) Each of the following row reduced matrices are augmented matrices coming from a system of linear equations. Decide in each case whether the corresponding system has one, zero or infinitely many solutions and put a check mark in the appropriate column.

Matrix	0 solution	1 solution	infinitely many solutions
$\left[\begin{array}{cccc c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right]$			

Matrix	0 solution	1 solution	infinitely many solutions
$\left[\begin{array}{cccc c} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right]$			

Matrix	0 solution	1 solution	infinitely many solutions
$\left[\begin{array}{cccc c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$			

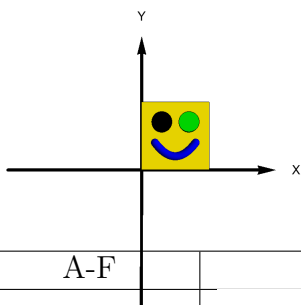
b) (4 points) Mark the sets which are linear spaces

☐	The set of polynomials f in P_2 for which $f(2) = 3$.
☐	The set of functions f in C for which $f(2) = f(3)$.
☐	The set of vectors $\vec{x} = (x, y, z)$ in three dimensional space for which $x - y = z$
☐	The kernel of the matrix $A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

c) (3 points) Exactly one of the following matrices is a reflection, exactly one of the matrices is rotation by 90° and exactly one of them is a projection onto a subspace. Decide which is which.

Matrix	is a reflection	is a rotation by 90°	is a projection
$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$			
$\begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}$			
$\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$			

Problem 3) (10 points) No justifications are necessary.



a) (8 points) Each figure illustrate the action of a transformation $T(x) = Mx$, where M is a 2×2 matrix. The picture to the left shows the case $M = I_2$. Match them with the matrices A) – F). There is a one to one match.

A-F		A-F	

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad E = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \quad F = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$$

b) (2 points) One of the following formulas is always true if A, B, C are arbitrary invertible 7×7 matrices and $AXB = C$. Which one?

$X = CA^{-1}B^{-1}$	
$X = CB^{-1}A^{-1}$	
$X = A^{-1}CB^{-1}$	
$X = A^{-1}B^{-1}C$	
$X = B^{-1}A^{-1}C$	

Problem 4) (10 points)

Let's look at the system of equations

$$x + y + z + w = 10$$

$$x - y + z - w = 2$$

$$x + y - z - w = 0$$

a) (3 points) Write the system in matrix form $A\vec{x} = \vec{b}$.

b) (7 points) Find all the solutions of the system.

Problem 5) (10 points)

Find a basis for the linear space V of all vectors in R^5 perpendicular to $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. As

usual, we want to see details of the computation.

Problem 6) (10 points)

The **2048 game** deals with 4×4 matrices. A couple of years ago, it became a viral hit. Oliver never succeeded in reaching 2048 and instead invented the “anti 2048 game”, where the goal is to end the game with the least possible score. In the following matrix A , the coefficients of the matrix sum up to 96 points:

2	4	8	4
4	8	4	2
	2	16	32
	2	2	16

$$A = \begin{bmatrix} 2 & 4 & 8 & 4 \\ 4 & 8 & 4 & 2 \\ 2 & 4 & 8 & 4 \\ 4 & 8 & 4 & 2 \end{bmatrix}$$

a) (5 points) Find a basis for the kernel of A . (As usual, even if you should see the answer, we want you to derive the answer).

b) (5 points) Find a basis for the image of A . (Also here, we want you to tell how you got the answer).

Problem 7) (10 points)

a) (3 points) Find an orthonormal basis $\mathcal{B}$ of $\mathbf{R}^3$, which contains the two vectors

$$\vec{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

b) (7 points) Find the 3×3 matrix belonging to the transformation T which reflects first about the line spanned by the vector $\vec{v}_1$ and then reflects about the line containing the vector $\vec{v}_2$.

Problem 8) (10 points)

Computer vision and **relativity** provide reasons why transformations in 4 dimensions can be important in applications. We look here at computer vision, where we can implement more general transformations by increasing the dimension.

a) (3 points) Decide whether the transformation

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$$

is a linear transformation or not. In any case, give a reason.

b) (3 points) Decide whether the transformation

$$S\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 1 & 0 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

is a linear transformation or not. In any case, give a reason.

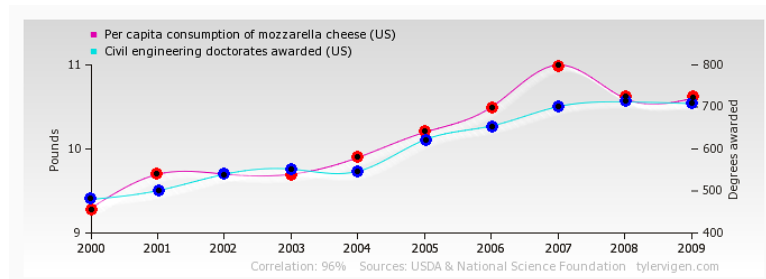
c) (4 points) Verify that the first three coordinates of $S\left(\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}\right)$ agree with the three coordinates of $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$.

Problem 9) (10 points)

That **correlation does not necessarily imply causation** is one of the most important insights, when analyzing data. One has for example found a strong correlation between per capita consumption of **mozzarella cheese** x and the number of **civil engineering doctorates** y

awarded in the US. Assume the data are given by the vectors $x = \begin{bmatrix} 9 \\ 10 \\ 14 \end{bmatrix}$ and $y = \begin{bmatrix} 500 \\ 600 \\ 700 \end{bmatrix}$.

In the real data, the correlation has been computed to be 0.96 with 10 data points. Here we only take three data points.



a) (6 points) Find the vectors with coordinates $X_i = x_i - E[x]$, $Y_i = y_i - E[y]$, where $E[x] = (x_1 + x_2 + x_3)/3$ is the **expectation**. The vectors X, Y have now zero expectation. Also compute the dot product $X \cdot Y$, as well as the lengths $|X|, |Y|$. This corresponds up to a normalization to the **covariance** $\text{Cov}[X, Y] = X \cdot Y/3$ and **standard deviations** $\sigma[X] = |X|/\sqrt{3}$, $\sigma[Y] = |Y|/\sqrt{3}$ of the random variables X, Y in statistics.

b) (4 points) Find the **correlation coefficient**

$$\frac{X \cdot Y}{|X||Y|} = \frac{\text{Cov}[X, Y]}{\sigma[X]\sigma[Y]}$$

which we know to be the cosine of the angle between X and Y .