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- Please fill in your name and mark your section.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) (20 points) True or False? No justifications are needed.

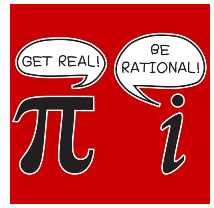
1)	T	F	If A is the matrix representing a shear in the plane then $\det(A - I_2) = 0$.
2)	T	F	Two matrices are similar if and only if they have the same eigenvalues.
3)	T	F	Every orthogonal $n \times n$ matrix satisfies $A^2 = I_n$.
4)	T	F	The eigenvalues of a $n \times n$ matrix A do not change under row reduction.
5)	T	F	Every 2×2 rotation matrix can be diagonalized over the complex numbers.
6)	T	F	The recursion $x(t+1) = x(t)^2 - x(t-1)^2$ can be written as a vector equation $\vec{v}(t+1) = A\vec{v}(t)$ for a 2×2 matrix A and a vector $\vec{v}$.
7)	T	F	The product of two rotations in the plane can be a reflection at a line.
8)	T	F	The rank of an $n \times n$ matrix A is the same as the rank of A^T.
9)	T	F	For any 3×3 matrix, we have $\det(A^3/3) = \det(A)^3/27$.
10)	T	F	A discrete dynamical system is asymptotically stable if the absolute value of the trace A and determinant are both smaller than 1.
11)	T	F	A reflection dilation in the plane has zero trace.
12)	T	F	The matrix $A = \begin{bmatrix} 7 & 0 \\ 5 & 3 \end{bmatrix}$ is similar to $B = \begin{bmatrix} 3 & 0 \\ 5 & 7 \end{bmatrix}$.
13)	T	F	If two 4×4 matrices have each the eigenvalues 1 with algebraic multiplicity 4 and the same geometric multiplicity 2, then they are similar.
14)	T	F	If A has only the trivial kernel so that A^TA is invertible then the least square solution of $Ax = b$ is unique.
15)	T	F	If $\vert A^{-n}\vec{v}\vert$ goes to infinity for $n \to +\infty$ for every non-zero $\vec{v}$, then $\vert A^n\vec{v}\vert$ goes to zero for $n \to +\infty$ for every $\vec{v}$.
16)	T	F	There is an orthogonal anti-symmetric matrix. (Remember that anti-symmetric=skew symmetric means $A^T = -A$).
17)	T	F	If $A = QR$ is the QR decomposition of a square matrix, then A is stable if and only if R is stable.
18)	T	F	For every 2×2 matrix A, we know that A is similar to A^{-1}.
19)	T	F	If an invertible A is similar to B then $A^{10} + A^{-1}$ is similar to $B^{10} + B^{-1}$.
20)	T	F	If A is a symmetric 2×2 matrix, has trace 2 and determinant 1, then it is the identity.

	Total

Problem 2) (10 points) No justifications are needed.

a) (2 points) Which of the following complex numbers are real?

Number	is real	is not real
$z = e^{i\pi}$		
$z = 1/i$		
$z = i^4 - i^2$		
$z = e^{i\pi/2} - i$		



b) (2 points) Which matrices have the property that the system $x(t+1) = Ax(t)$ is stable?

Matrix	stable	not stable
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$		
$\begin{bmatrix} 0.8 & 1 \\ 0 & -0.7 \end{bmatrix}$		
$\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$		
$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$		

c) (2 points) Which identities do always hold?

Identity	always true	not always true
$\det(A^2 - B^2) = \det(A^2) - \det(B^2)$		
$\det(A^2 - B^2) = \det(A - B)\det(A + B)$		
$\mathrm{tr}(A^2 - B^2) = \mathrm{tr}(A^2) - \mathrm{tr}(B^2)$		
$\mathrm{tr}(A^2 - B^2) = \mathrm{tr}(A - B)\mathrm{tr}(A + B)$		

d) (2 points) Which of the following formulas define the projection onto the linear space $V = \mathrm{im}(A)$, if A has no kernel?

Formula	is a projection onto V	is not a projection onto V
AA^T		
$A(A^TA)^{-1}A^T$		
$(A^TA)^{-1}A^T$		
$(AA^T)^{-1}A^T$		

e) (2 points) Assume S is a matrix which contains an eigenbasis of an $n \times n$ matrix A as columns. Which of the following statements are true?

Statement	is always true	is not always true
SAS^{-1} is diagonal		
$S^{-1}AS$ is diagonal		
A is diagonal		
S is diagonal		

Problem 3) (10 points) No justifications are needed

a) (2 points) In the following, A is a 2×2 matrix of a given type.

Type	$\det(A) = 1$	A^2 is same type	always real eigenvalues	diagonalizable
Rotation				
Projection				
Symmetric				
Reflection				

b) (3 points) Assume v is an eigenvector to A with eigenvalue 3 and A is invertible. Then

Statement	True	False
A^5 has an eigenvalue 3^5		
A^{-1} has an eigenvalue $1/3$		
A^T has an eigenvalue $1/3$		
A^5 has an eigenvector v		
A^{-1} has an eigenvector v		
A^T has an eigenvector v		

c) (3 points) Which of the following statements are true about a real eigenvalue λ of a real matrix A?

Statement	True	False
If $A = A^T$ then the geometric and algebraic multiplicities of λ agree		
If $A \neq 1$ is a shear, then the geometric and algebraic multiplicity of λ do not agree		
The geometric multiplicity of λ is always positive		
The algebraic multiplicity of λ is always positive		
The algebraic multiplicity of λ can be smaller than the geometric multiplicity		
If $x(t+1) = Ax(t)$ is stable and B is similar to A then $x(t+1) = Bx(t)$ is stable		

d) (2 points) Which statements are true for real matrices A, B?

Statement	True	False
If A is similar to B and A is symmetric then B is symmetric		
A diagonal matrix A is always symmetric		
A symmetric matrix A is always diagonalizable		
A symmetric matrix A always has real eigenvalues		

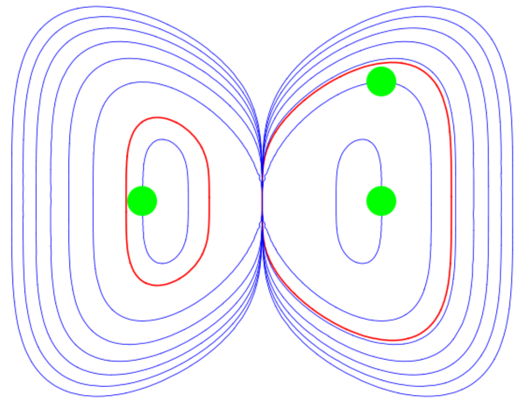
Problem 4) (10 points)

We want to model some data points with a quartic curve of the form

$$x^4 + y^4 + ax^2 - bx = 0 \ ,$$

where a, b are unknown parameters. Find the best linear fit to the following data points:

$$\begin{array}{rcl} (x, y) & = & (1, 1) \\ (x, y) & = & (-1, 0) \\ (x, y) & = & (1, 0) \ . \end{array}$$



Problem 5) (10 points)

The sequence $1, 1, 4, 7, 16, 31, ..$ is obtained from the recursion $x(t+1) = x(t) + 2x(t-1) + 1$. This can be written as a discrete dynamical system $v(t+1) = Av(t)$, where

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, v(t) = \begin{bmatrix} x(t) \\ x(t-1) \\ 1 \end{bmatrix} \ .$$

a) (3 points) Knowing that $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$ are eigenvectors, find the eigenvalues of A.

b) (7 points) The initial condition $\vec{v}(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ gives us the above sequence. Write down the explicit expression for $v(t) = A^t v(0)$.

Problem 6) (10 points)

Decide in the following cases whether the dynamical system $v(t+1) = Av(t)$ is asymptotically stable.

a) (2 points) $A = \begin{bmatrix} 2 & 3 & 2 \\ 1 & 1 & 2 \\ 4 & 1 & 0 \end{bmatrix}$.

b) (2 points) $A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 2 & 0 \\ 5 & 0 & 0 \end{bmatrix}$.

c) (2 points) $A = \begin{bmatrix} 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 \end{bmatrix}$.

d) (2 points) Why is the following rule true for a 3×3 matrix: if A is stable then $|\det(A)| < 1$.

e) (2 points) Why is the following rule true for a 3×3 matrix: if A is stable then $|\text{tr}(A)| < 3$.

Problem 7) (10 points)

a) (2 points) Find the determinant of the matrix

$$\begin{bmatrix} 9 & 2 & 2 & 2 & 2 & 2 \\ 1 & 8 & 1 & 1 & 1 & 1 \\ 1 & 1 & 8 & 1 & 1 & 1 \\ 1 & 1 & 1 & 8 & 1 & 1 \\ 1 & 1 & 1 & 1 & 8 & 1 \\ 2 & 2 & 2 & 2 & 2 & 9 \end{bmatrix}.$$

b) (3 points) Find the determinant of the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 & 0 & 0 & 0 \\ 4 & 4 & 4 & 4 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

c) (2 points) Find the determinant of the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

d) (3 points) Find the determinant of the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \\ 4 & 5 & 1 & 2 & 3 \\ 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 \end{bmatrix}$$

Problem 8) (10 points)

Find the possibly complex eigenvalues and eigenbasis for the following matrices:

a) (2 points)

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

b) (2 points)

$$B = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$$

c) (2 points)

$$C = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$$

d) (2 points)

$$D = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$$

e) (2 points)

$$E = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}.$$

Problem 9) (10 points)

Find the QR decomposition of the same matrices as in the previous problem (Déjà-vu):

a) (2 points)

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

b) (2 points)

$$B = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$$

c) (2 points)

$$C = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$$

d) (2 points)

$$D = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$$

e) (2 points)

$$E = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} .$$

Problem 10) (10 points)

Let $J = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$.

a) (2 points) Is J orthogonal?
b) (1 point) Is it symmetric?
c) (2 points) What is the determinant?
d) (3 points) A 4×4 matrix A is called **symplectic**, if $A^T J A = J$. Verify that the determinant of A must be either 1 or -1.
e) (2 points) Verify that the inverse of a symplectic matrix A is given by

$$J^T A^T J .$$

Remark: symplectic matrices are important in physics, especially in celestial mechanics.