Name:

MWF 9 Fabian Haiden
MWF 10 Ziliang Che
MWF 10 Jeremy Hahn
MWF 11 Rosalie Belanger-Rioux
MWF 11 Yu-Wen Hsu
MWF 12 Peter Garfield
TThu 10 Oliver Knill
TThu 11:30 Alex Perry
TThu 11:30 Rong Zhou

- Please fill in your name and mark your section.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) (20 points) True or False? No justifications are needed.

1) T **F** A 4×4 matrix whose entries are all 4 has determinant 4^4.

Solution:
The determinant is zero because the matrix has identitcal rows.

2) T **F** $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is an orthogonal matrix.

Solution:
It is an orthogonal **projection**, not an orthogonal **transformation**.

3) T **F** Every 2×2 matrix is diagonalizable over the complex numbers.

Solution:
The shear is a counter example. It can not be diagonalized even using complex numbers. There is an eigenvector to the eigenvalue 1 with geometric multiplicity 1.

4) **T** F If A is a projection onto a linear subspace V, then $\text{im}(A) = V$ and $\ker(A) = V^\perp$.

Solution:
The image is clearly V. Everything perpendicular to the image is maped to the origin. One can see second part also using the formula $\ker(A) = (\text{im}(A^T))^\perp$ and the fact that $A = A^T$ for a projection.

5) **T** F If A is a 3×3 matrix representing reflection about a line in $\mathbb{R}^3$, then A is symmetric.

Solution:
The projection P onto a line is symmetric. Because the reflection R satisfies $R = 2P - I$, also the reflection $R = 2P - I$ is symmetric.

6) **T** F If A is a matrix with orthonormal columns, then $\vec{x} = A^T\vec{b}$ must be a least-squares solution of the system $A\vec{x} = \vec{b}$.

Solution:
Because $A^T A$ is the identity matrix 1_n where n is the number of columns of A, the formula $(A^T A)^{-1} A^T \vec{b}$ simplifies to $A^T \vec{b}$.

7) T [highlighted] | F

If A represents the projection onto a line in $\mathbb{R}^2$, then A is similar to $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Solution:
Yes, a projection is diagonalizable. It has one eigenvalue 1 and one eigenvalue 0.

8) T | F [highlighted]

If A is a square matrix such that $A\vec{v} \cdot A\vec{w} = 0$, whenever $\vec{v} \cdot \vec{w} = 0$, then A is an orthogonal matrix.

Solution:
Take the transformation which scales by a factor 2. It preserves angles but not lengths and is not an orthogonal transformation. Nevertheless, it preserves right angles.

9) T | F [highlighted]

If A is a 2×2 matrix with the eigenvalues 0 and 1, then A is the matrix of an orthogonal projection.

Solution:
It is a projection but not necessarily an orthogonal projection.

10) T | F [highlighted]

If A and B are similar, then they have the same eigenvectors.

Solution:
They have the same **eigenvalues**, but not necessarily the same eigenvectors. The shear for example has the eigenvector e_1, while its tanspose has the eigenvector e_2.

11) T | F [highlighted]

If A and B are both diagonalizable, then AB is diagonalizable.

Solution:
Not necessarily. there are $2x2$ matrices A, B for which the statement is not true: example: $A = \begin{bmatrix} 2 & 3 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ 0 & 2 \end{bmatrix}$ are both diagonalizable, but $AB = \begin{bmatrix} 6 & 10 \\ 0 & 6 \end{bmatrix}$ is not.

12) T **F** If λ is an eigenvalue of A and μ is an eigenvalue of B, then $\lambda\mu$ is an eigenvalue of AB.

Solution:
This is even not true for all diagonal 2×2 matrices with different eigenvalues. Take $A = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$. The number 15 is a product of eigenvalues of A and B but not an eigenvalue of AB.

13) **T** F If $S^{-1}AS$ is a diagonal matrix, then the columns of S must be eigenvectors of A.

Solution:
Yes, S is then the conjugation to the diagonal matrix and $S^{-1}ASe_i = e_i$ means $ASe_i = Se_i$ so that Se_i is an eigenvector. But Se_i is the i'th column of S.

14) T **F** If λ is an eigenvalue of a 2×2 matrix A, then $-\lambda$ is an eigenvalue of A.

Solution:
Take a diagonal matrix with eigenvalues $2, 3$.

15) T **F** If A is a 5×5 matrix, then $\det(5A) = 25\det(A)$.

Solution:
We have $\det(5A) = 5^5\det(A)$.

16) T **F** Any two 2×2 matrices whose eigenvectors are equal must be similar.

Solution:
Two different diagonal matrices have the same eigenvectors but are not similar

17) **T** F The standard basis vectors of $\mathbf{R}^n$ are the eigenvectors of every diagonal $n \times n$ matrix.

Solution:
Obvious, in the k'th row, we have a multiple of the k'th basis vector.

18) [T] [ ] If $A^2 = A$, then every eigenvalue λ of A is either $\lambda = 1$ or $\lambda = 0$.

T (highlighted) | F

Solution:
If A has an eigenvalue λ, then for the corresponding eigenvector $\vec{v}$ we have $A^2\vec{v} = \lambda^2\vec{v}$ and $A\vec{v} = \lambda\vec{v}$. So, $\lambda^2 = \lambda$.

19) T (highlighted) | F

If A is a symmetric matrix with characteristic polynomial $(1-\lambda)^2(2-\lambda)$, then the 1-eigenspace of A is the orthogonal complement of the 2-eigenspace of A.

Solution:
Eigenvectors of a symmetric matrix are perpendicular

20) T | F (highlighted)

A 2×2 matrix with characteristic polynomial $f_A(\lambda) = (2-\lambda)^2$ is similar to either $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Solution:
It would be consequence of the Jordan normal form theorem, treated in class if the characteristic polynomial were $(1-\lambda)^2$. But the characteristic polynomial here does not match.

Problem 2) (10 points)

Check the boxes, where the two matrices A and B are similar. No explanations are necessary for this problem. Each pair counts 2 points.

a) $\square$ $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

b) $\square$ $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

c) $\square$ $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

d) $\square$ $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

e) $\square$ $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Solution:
Only b) and c) are similar. a) are different Jordan normal forms, b) are similar because A is diagonalizable where it is B), c) can be conjugated by a permutation matrix, d) have different eigenvalues, for e) one can see $A^2 = 0$, while $B^2 \neq 0$.

Problem 3) (10 points)

Consider the matrix $A = \begin{bmatrix} 5 & 1 & 1 & 1 & 1 & 1 \\ 1 & 5 & 1 & 1 & 1 & 1 \\ 1 & 1 & 5 & 1 & 1 & 1 \\ 1 & 1 & 1 & 5 & 1 & 1 \\ 1 & 1 & 1 & 1 & 5 & 1 \\ 1 & 1 & 1 & 1 & 1 & 5 \end{bmatrix}$.

a) Find the kernel of $B = A - 4I_6$.

b) Show that $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of A. What is the eigenvalue?

c) Find all the eigenvalues of A with their algebraic multiplicities.

d) Write down the eigenbasis of A. What are the geometric multiplicities of the eigenvalues?

e) What is $\det(A)$?

Solution:

a) The kernel of $B = A - 4I = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$ is the kernel of its row reduced echelon form $\text{rref}(B) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. which is spanned by the 5 vectors

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

b) The vector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of B to $\lambda = 6$. So, it is an eigenvector of A with eigenvalue $\lambda = 10$.

c) $B = A - 4I_6$ has the eigenvalue 6 with multiplicity 1 and the eigenvalue 0 with multiplicity 5. So A has the eigenvalue 10 with multiplicity 1 and the eigenvalue 4 with multiplicity 5.

d) The eigenvectors of $A - 4I_6$ are the same as the eigenvectors of A. because if $A\vec{v} = \lambda\vec{v}$, then $(A - 4I_6)\vec{v} = (\lambda - 4)\vec{v}$.

e) The determinant is the product of the eigenvalues which is $4^5 \cdot 10 = \boxed{10'240}$.

Problem 4) (10 points)

Find the function $y = f(x) = a + b2^x$, which best fits the data

x	y
0	1
1	3
2	7

Solution:
We have to find the least square solution to the system of equations

$$\begin{aligned} a + b &= 1 \\ a + 2b &= 3 \\ a + 4b &= 7 \end{aligned}$$

which is in matrix form written as $A\vec{x} = \vec{b}$ with

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix} .$$

Now $A^T\vec{b} = \begin{bmatrix} 11 \\ 35 \end{bmatrix}$ and $A^TA = \begin{bmatrix} 3 & 7 \\ 7 & 21 \end{bmatrix}$ and $(A^TA)^{-1} = \begin{bmatrix} 21 & -7 \\ -7 & 3 \end{bmatrix}/14$ and $(A^TA)^{-1}A^T\vec{b}$ is $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$. The best fit is the function $\boxed{f(x) = -1 + 2 \cdot 2^x}$.
Actually, the minimal fit produces an actual solution through the data.

Problem 5) (10 points)

Let $A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 2 & -1 \\ 6 & 0 & -4 \end{bmatrix}$. You are told that $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an eigenbasis for A, where:

- $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ belongs to the eigenvalue -1,
- $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ belongs to the eigenvalue 2,
- $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ belongs to the eigenvalue -2.

Find an eigenbasis for A^T.

Solution:
There are two possibilities to solve this problem. Since we know the eigenvalues, we just have to find the new eigenvectors to A^T. They are

- $\vec{v}_1 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ belongs to the eigenvalue -1,
- $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ belongs to the eigenvalue 2,
- $\vec{v}_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ belongs to the eigenvalue -2.

The second solution is to note that $S^{-1}AS = B$ implies $S^T A^T (S^{-1})^T = B$ so that we can obtain the eigenvectors as the columns of $(S^{-1})^T$, where

$$S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

is the matrix containing the eigenvectors of A as column vectors. We have

$$(S^{-1})^T = \begin{bmatrix} 3 & -1 & -2 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

and can read off the eigenvectors of A^T as the column vectors of this matrix.

Problem 6) (10 points)

a) (4 points) Find the Gram-Schmidt orthogonalization of the basis $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$.

b) (4 points) Find the QR decomposition of the matrix $A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

c) (2 points) What is the volume of the parallelepiped spanned by the column vectors of A?

Solution:
a) The first vector $\vec{v}_1$ is already normalized so that $\vec{w}_1 = \vec{v}_1$.

$\vec{u}_2 = \vec{v}_2 - (\vec{w}_1 \cdot \vec{v}_2)\vec{w}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ This is already normalized.

$\vec{u}_3 = \vec{v}_3 - (\vec{w}_1 \cdot \vec{v}_3)\vec{w}_1 - (\vec{w}_2 \cdot \vec{v}_3)\vec{w}_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ which is normalized equal to $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

b) $Q = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $R = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$.

c) The volume is the absolute value of the determinant of R which is $\boxed{2}$. (You could also see directly that the volume is $|\det(A)| = 2$.)

Problem 7) (10 points)

a) Find the determinant of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 & 5 \\ 3 & -2 & -3 & -4 & 0 \end{bmatrix}$$

b) Find the determinant of the matrix

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 & 2 & 3 \end{bmatrix}$$

Solution:
a) You can use Laplace to make the expansion. But it is easier to see that two rows are identical so that the matrix is not invertible. Therefore $\boxed{\det(A) = 0}$.
b) This is a partitioned matrix having 3 matrices of size 2×2 in the diagonal. The determinant of A is $\det \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \cdot \det \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \cdot \det \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} = (-3) \cdot 1 \cdot 1 = \boxed{-3}$.

Problem 8) (10 points)

To support their teams, Harvard and Yale fans both sell T-shirts in November 2004. Each day, the following happens:

- Every Harvard fan recruits two more Harvard fans and four Yale fans.
- Every Yale fan recruits two more Yale fans and one Harvard fan.

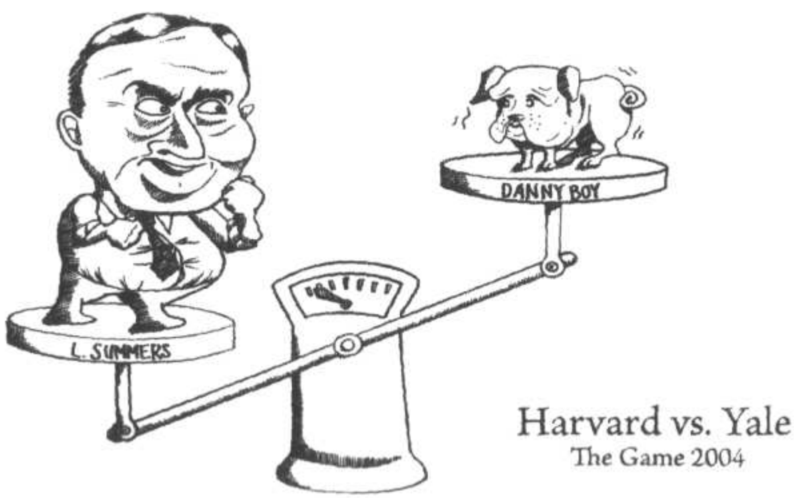
Let $H(t)$ and $Y(t)$ be the number of Harvard fans and Yale fans, respectively, on day t. We can model the above situation with a dynamical system

$$\begin{bmatrix} H(t+1) \\ Y(t+1) \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} H(t) \\ Y(t) \end{bmatrix}.$$

Suppose that $H(0) = Y(0) = 100$.

a) (5 points) Find a closed formula for $H(t), Y(t)$.

b) (5 points) What happens with $H(t)/Y(t)$ for large t?



Solution:

We have to find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 3 & 4 \\ 1 & 3 \end{bmatrix}$.

The eigenvector to the eigenvalue 5 is $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, the eigenvector to the eigenvalue 1 is $\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Because $(75\vec{v}_1 + 25\vec{v}_2)$ is the initial condition, we know the solution is

$$A^n\vec{v} = A^n(75\vec{v}_1 + 25\vec{v}_2) = 75 \cdot 5^n\vec{v}_1 + 25 \cdot 1^n \cdot \vec{v}_2 = 75 \cdot 5^n \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 25 \cdot 1^n \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

b)

$$\frac{H(t)}{Y(t)} = \frac{75 \cdot 5^n \cdot 2 - 50 \cdot 1^n \cdot 2}{75 \cdot 5^n \cdot 1 + 25 \cdot 1^n \cdot 1} \to 2$$

c) No, it is $\boxed{\text{not stable}}$. Stability was defined as the property that $A^n\vec{x} \to \vec{0}$ for all vectors $\vec{x}$ and happens if all eigenvalues have absolute value $|\lambda| < 1$. We have here two eigenvalues which violate this. Indeed, for most initial conditions, $|A^n\vec{x}|$ converges to ∞ like for the ininitial condition $\vec{x} = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$.

Problem 9) (10 points)

Our goal is to find the determinant of

$$A = \begin{bmatrix} 101 & 1 & 1 & 1 & 1 & 1 \\ 2 & 102 & 2 & 2 & 2 & 2 \\ 3 & 3 & 103 & 3 & 3 & 3 \\ 4 & 4 & 4 & 104 & 4 & 4 \\ 5 & 5 & 5 & 5 & 105 & 5 \\ 6 & 6 & 6 & 6 & 6 & 106 \end{bmatrix}$$

a) The matrix $A - 100I_6$ has an eigenvalue 0. Find its algebraic multiplicity.

b) The matrix A^T has an eigenvector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Find the corresponding eigenvalue λ.

c) Why does A also have the same eigenvalue λ?

d) You have found all the eigenvalues of $A - 100I_6$. What are the eigenvalues of A?

e) Find the determinant of A.

Solution:

a) The matrix

$$B = A - 100I_6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 & 5 & 5 \\ 6 & 6 & 6 & 6 & 6 & 6 \end{bmatrix}$$

has identical columns and is therefore not invertible. It must have an eigenvalue 0. Row reduction gives

$$\text{rref}(B) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which shows that the image is 1 dimensional and the kernel is 5 dimensional. Because the geometric multiplicity is 5, the algebraic multiplicity is 5 or larger. Because the trace is not zero, the algebraic multiplicity can not be 6. The trace is the sum of the eigenvalues would be 0 if the algebraic multiplicity of 0 would be 6.

b) B^T has an eigenvalue $\lambda = 121$. Because B^T and B have the same eigenvalues (they have the same characteristic polynomial because transposed matrices have the same determinant), also B has the eigenvalue 121.

c) $A = B + 100I_6$ has the eigenvalue $21 + 100$ with algebraic multiplicity 1 and $0 + 100$ with algebraic multiplicity 5.

d) The determinant of A is the product of the eigenvalues, which is $\boxed{100^5 \cdot 121}$.

Problem 10) (10 points)

We want to find a formula for the general term x_n in the recursion

$$x_{n+1} = x_n + 3x_{n-1}/4$$

if $x_0 = 0, x_1 = 1$. This is the case of a Fibonacci recursion, in which only 3/4 of the previous generation has kids.

a) Write the recursion in the form $v_{n+1} = Av_n$ for vectors $v_n = \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}$ in the plane.

b) Find the eigenvalues λ_+, λ_- and eigenvectors v_+, v_- of A.

c) Write $v_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as $v_0 = av_+ + bv_-$.

d) Find $v_n = A^n v_0$ and so x_n.

Solution:
a) $A = \begin{bmatrix} 1 & 3/4 \\ 1 & 0 \end{bmatrix}$. b) The characteristic polynomial is $f_A(\lambda) = x^2 - x - 3/4$ which has the roots $\lambda_+ = 3/2$ and $\lambda_- = -1/2$. These are the eigenvalues.
c) The eigenvectors are

$$v_+ = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

and

$$v_- = \begin{bmatrix} -1 \\ 2 \end{bmatrix} .$$

d) $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = v = v_+/4 - v_-/4$. Therfore $A^n v = \lambda_+^n v_+ + \lambda_-^n v_-$. Written out, this is

$$A^n \langle [\begin{matrix} 1 \\ 0 \end{matrix} \rangle] = \frac{3^n}{4 \cdot 2^n} \begin{bmatrix} 3 \\ 2 \end{bmatrix} - \frac{(-1)^n}{4 \cdot 2^n} \begin{bmatrix} -1 \\ 2 \end{bmatrix} .$$

Therefore, $\boxed{x_n = \frac{3^n - (-1)^n}{2^{n+1}}}$.