Name:

MWF 9 Oliver Knill
MWF 10 Jeremy Hahn
MWF 10 Hunter Spink
MWF 11 Matt Demers
MWF 11 Yu-Wen Hsu
MWF 11 Ben Knudsen
MWF 11 Sander Kupers
MWF 12 Hakim Walker
TTH 10 Ana Balibanu
TTH 10 Morgan Opie
TTH 10 Rosalie Belanger-Rioux
TTH 11:30 Philip Engel
TTH 11:30 Alison Miller

- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) (20 points) True or False? No justifications are needed.

1) T F The matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ has at least one real positive eigenvalue.

2) T F Every 3×2 matrix has a non-zero vector in its kernel.

3) T F For any 3×3 matrix A satisfying $A^2 = 0$, the image of A is a subspace of the kernel of A.

4) T F If an invertible 2×2 matrix A is diagonalizable then the matrix A^{-2} is diagonalizable.

5) T F The Fourier series of the function $f(x) = \sin(5x)$ is $\sin(5x)$.

6) T F The space of smooth functions satisfying $f(x) = \sin(x)f(-x)$ forms a linear space.

7) T F The geometric multiplicity of an eigenvalue 0 of A is equal to the nullity of A.

8) T F Let A, B be arbitrary 3×3 matrices. The eigenvalues of $A+B$ are the sum of the eigenvalues of A and B.

9) T F A continuous dynamical system $x'(t) = Ax(t)$ with a 2×2 matrix A is asymptotically stable if $\mathrm{tr}(A) < 0$.

10) T F If $\vec{x}(t+1) = A\vec{x}(t)$ is an asymptotically stable dynamical system, then each real eigenvalue λ of A satisfies $\lambda < 0$.

11) T F The sum of two reflection dilation matrices is a reflection dilation matrix.

12) T F The function $f(x,t) = \sin(3x)\cos(3t)$ solves the heat equation $f_t = f_{xx}$.

13) T F If a system of linear equations $A\vec{x} = \vec{c}$ with 2×2 matrix A has 2 different solutions, there exists $\vec{b}$ such that $A\vec{x} = \vec{b}$ has no solution.

14) T F The equilibrium point $(0,0)$ of the nonlinear system $x' = -2x$, $y' = -3y^3 - y$ is asymptotically stable.

15) T F All projection matrices are diagonalizable.

16) T F Using the length that comes from the Fourier series inner product, $||3\sin(5x) - 4\sin(10x)|| = 5$.

17) T F The product of the eigenvalues of a symmetric non-invertible 2×2 matrix is 0.

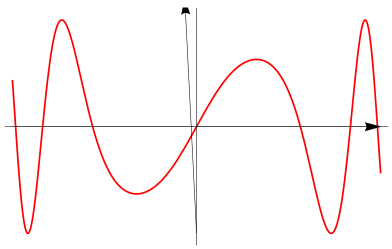
18) T F All solutions to the differential equation $x''(t) + 9x(t) = \sin(9t)$ stay bounded.

19) T F For any rotation matrix A, the transpose A^T is similar to A.

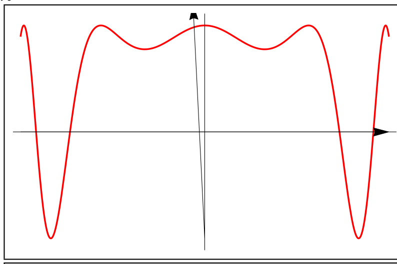
20) T F If the sum of the squares of the entries of a square matrix is 1, then A is invertible.

Problem 2) (10 points) No justifications needed

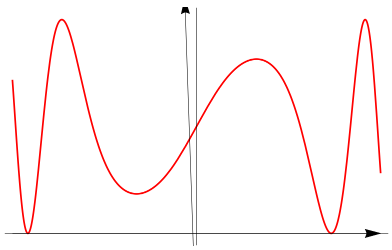
a) (3 points) The numbers a_n, b_n are Fourier coefficients. If the coefficient of f with respect to the constant function is zero, circle a_0, if the cos-coefficients a_n of f are all zero, circle a_n, if the sin-coefficients b_n of f are all zero, circle b_n.



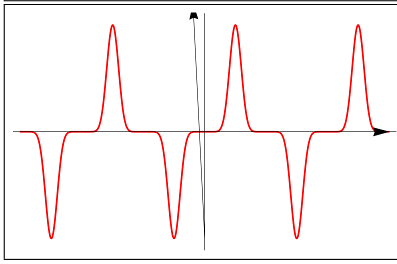
$\boxed{a_0}$ $\boxed{a_n}$ $\boxed{b_n}$



$\boxed{a_0}$ $\boxed{a_n}$ $\boxed{b_n}$



$\boxed{a_0}$ $\boxed{a_n}$ $\boxed{b_n}$



$\boxed{a_0}$ $\boxed{a_n}$ $\boxed{b_n}$

b) (3 points) Assume A is a $n \times n$ matrix for which

$$A^2 - A = 0 .$$

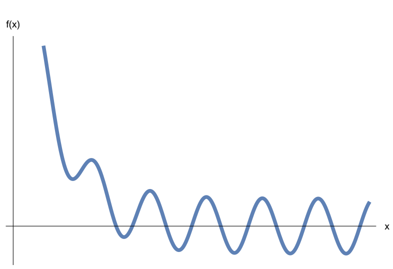
Check what is always true. We write O_n for the 0 matrix.

Statement	Always true
$A = 0_n$ or $A = I_n$	
A is not invertible	
At least one of the eigenvalue is 0 or 1	
A is a reflection	
A is orthogonal	
The trace of A is zero	

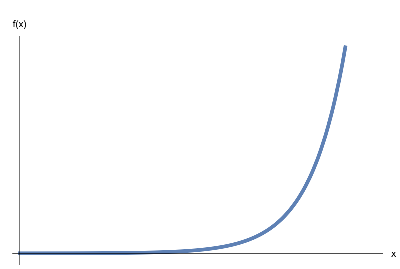
c) (4 points) Match the differential equations with possible solution graphs. There is an exact match.

Enter A-D	Differential equation
	$f'(t) - 16f(t) = 2\sin(4t)$
	$f''(t) + 16f(t) = 2\sin(16t)$

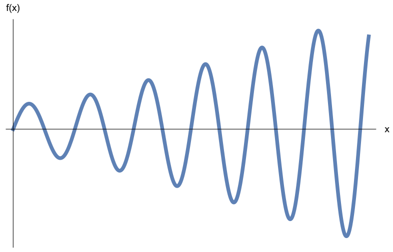
Enter A-D	Differential equation
	$f'(t) + 16f(t) = 2\sin(4t)$
	$f''(t) + 16f(t) = 2\sin(4t)$



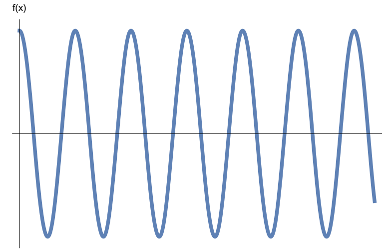
A)



B)



C)



D)

Problem 3) (10 points) No justifications needed

a) (4 points) Assume T is a transformation on C^{∞}_{per}, the linear space of 2π-periodic functions on the real line. Which T are linear?

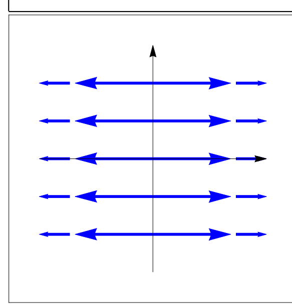
Transformation	Check if linear	Transformation	Check if linear
$Tf(x) = f(x+1)$		$Tf(x) = f(f(x)\cos(x))$	
$Tf(x) = f(\cos(x))$		$Tf(x) = \cos(f(x))$	
$Tf(x) = f'(x+\cos(x))$		$Tf(x) = \cos(x) + f(x)$	

b) (4 points) Match the differential equation

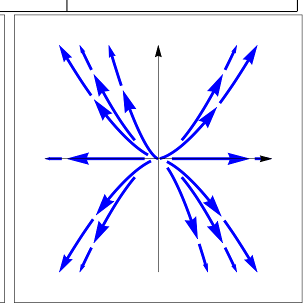
$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$$

with the phase portraits. There is an exact match.

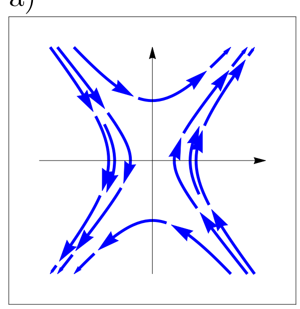
Matrix	Enter a) - d)
$A = \begin{bmatrix} 0 & 2 \\ -3 & 0 \end{bmatrix}$	
$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$	
$A = \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}$	
$A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$	



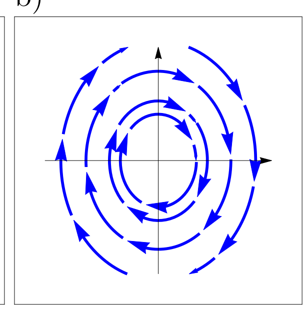
a)



b)

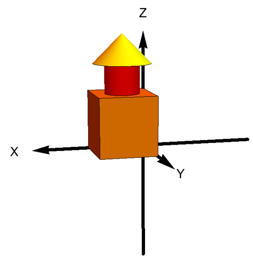


c)

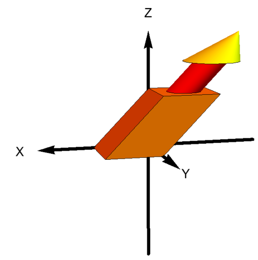


d)

c) (2 points) The following transformation T has been implemented by one of the matrices A,B,C or D. Which one?



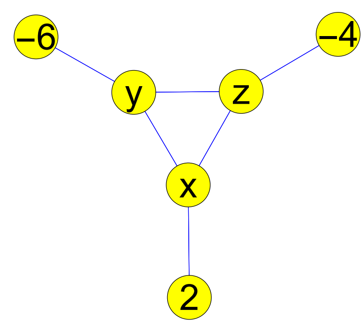
is mapped by T to



A= $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ B= $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ C= $\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ D= $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Problem 4) (10 points)

We look at a **social network**, where the values at nodes are opinions which can be both positive or negative. Some members have fixed opinions already and are not influenced by their friends. The rule is that the remaining people have the sum of the opinions of their friends. We still miss the opinions of three people x, y, z.



$$\left| \begin{array}{ccccc} x & - & y & - & z = 2 \\ -x & + & y & - & z = -6 \\ -x & - & y & + & z = -4 \end{array} \right|$$

a) (7 points) Write down the augmented 3×4 matrix $B = [A|b]$ which belongs to the system and find its row reduced echelon form. We ask you here to document which row reduction steps you do.

b) (3 points) What are the solutions to the system of linear equations?

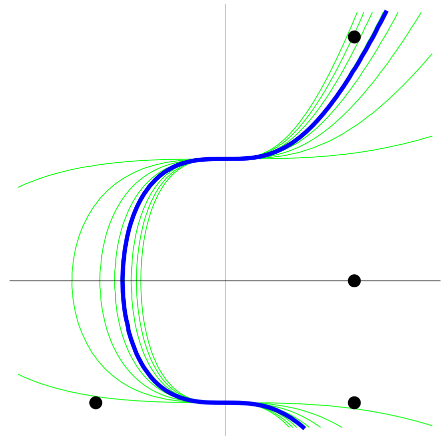
Problem 5) (10 points)

Using the least square method, find the best function

$$a + bx^3 = y^2$$

which best fits the data points (x, y):

$$\{(1,0),(1,2),(-1,-1),(1,-1)\ \}\ .$$



Problem 6) (10 points)

a) (6 points) Given the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \\ 4 & 8 & 12 & 16 \end{bmatrix}$$

Find a matrix S and a diagonal matrix B such that $S^{-1}AS = B$.

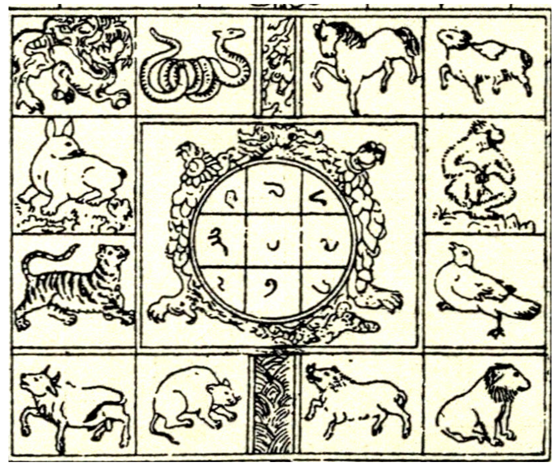
b) (4 points) Find the closed form solution of the discrete dynamical system

$$\vec{v}(t+1) = A\vec{v}(t)$$

with initial condition $\vec{v}(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.

Problem 7) (10 points)

According to the legend of **Lo Shu**, there had been a huge flood around 650 BC. As **king Yu** tried to channel the water to the sea, a turtle emerged with a 3×3 grid pattern on its shell. The numbers in each row, column and diagonal added up to 15, the number of days in each of the 24 cycles of the Chinese solar year. The pattern helped the king to control the river. The **Lo Shu square** is an example of a "magic square": $A = \begin{bmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{bmatrix}$.



a) (2 points) What is the sum of the eigenvalues of A?

b) (2 points) Find the eigenvalue of A to the eigenvector $[1,1,1]^T$.

c) (2 points) What is the determinant of A?

d) (2 points) The matrix A has two more eigenvalues $\pm i\sqrt{k}$, where k is an integer. Find k.

e) (2 points) The matrix $B = A/15$ is an example of a **double stochastic matrix**, a matrix for which all rows and columns add to 1. What are the eigenvalues of B?

Problem 8) (10 points)

In the Mathematica project we have looked at **Kirchhoff matrices**. Here is that matrix for the cyclic graph C_7:

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

a) (3 points) You see that $[1,1,1,1,1,1,1]^T$ is an eigenvector. What is the eigenvalue?

b) (3 points) Find the eigenvalues of A.

c) (4 points) Write down the eigenvectors of A.

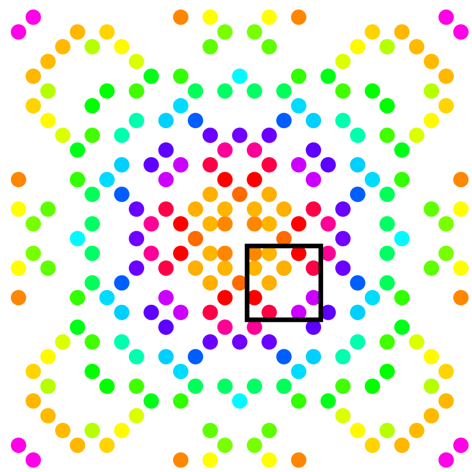
Problem 9) (10 points)

a) (4 points) Evaluate the determinant of

$$A = \begin{bmatrix} 18 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\ 7 & 17 & 7 & 7 & 7 & 7 & 7 & 7 \\ 6 & 6 & 16 & 6 & 6 & 6 & 6 & 6 \\ 5 & 5 & 5 & 15 & 5 & 5 & 5 & 5 \\ 4 & 4 & 4 & 4 & 14 & 4 & 4 & 4 \\ 3 & 3 & 3 & 3 & 3 & 13 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 & 2 & 12 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 11 \end{bmatrix}$$

b) (3 points) The matrix A is defined by **Gaussian primes** $A(n,m) = 1$ if $n - im$ is prime and $A(n,m) = 0$ if not. You don't need to know about Gaussian primes to find the determinant of

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$



In this problem we worked with the matrix A_5. The **21b conjecture** is: all matrices A_n are invertible for $n > 28$.

c) (3 points) Evaluate the determinant of the matrix

$$A = \begin{bmatrix} 8 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Problem 10) (10 points)

Solve the following differential equations:

a) (2 points)

$$f'''(t) = e^t$$

b) (2 points)

$$f'(t) + f(t) = e^t$$

c) (2 points)

$$f'(t) - f(t) = e^t$$

d) (2 points)

$$f''(t) - f(t) = e^t$$

e) (2 points)

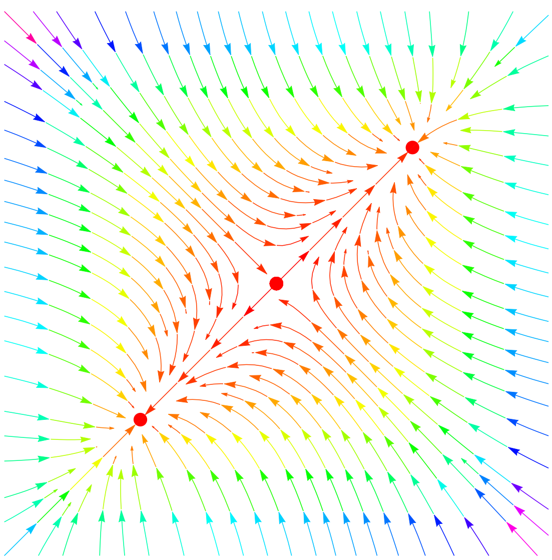
$$f''(t) + f(t) = e^t$$

Problem 11) (10 points)

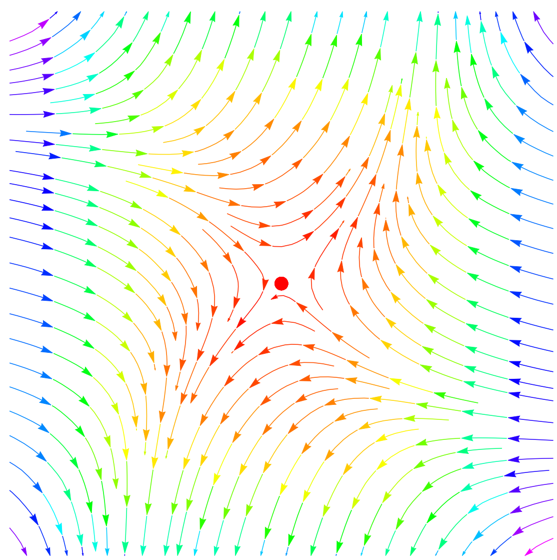
Find and analyze the stability of the equilibrium points of the nonlinear dynamical system

$$\begin{aligned} \frac{d}{dt}x &= y - x^3 \\ \frac{d}{dt}y &= x - y^3 \ . \end{aligned}$$

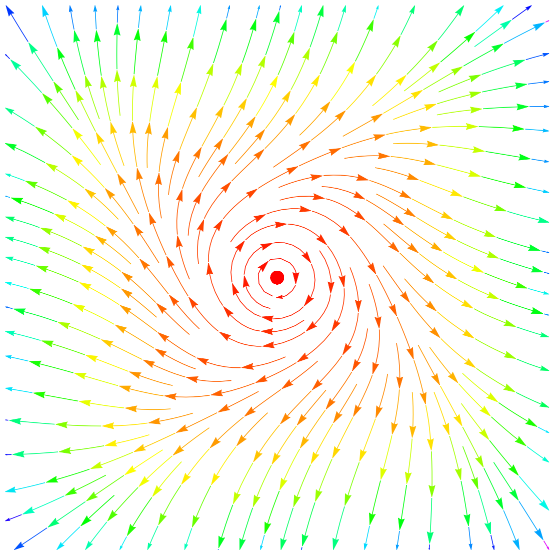
As usual, we ask you to find the **null clines**, the **equilibrium points** as well want you to perform a **linear stability analysis** at those points using the Jacobian. Which of the following four phase portraits fits the system?



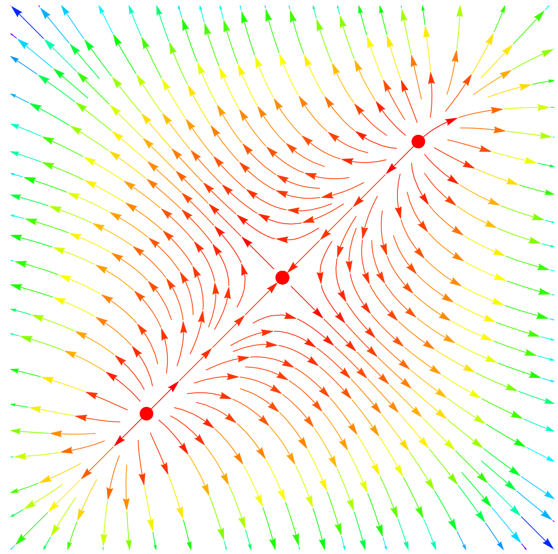
A



B



C



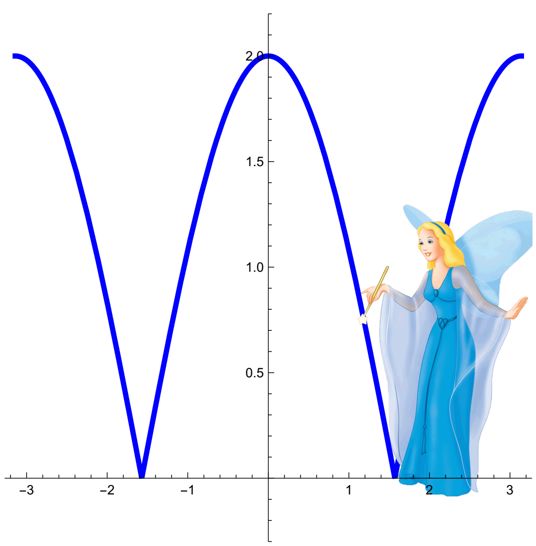
D

Problem 12) (10 points)

a) (6 points) Find the **Fourier series** of the function

$$f(x) = 2|\cos(x)| \; .$$

The graph of the function f on $[-\pi, \pi]$ is displayed to the right.



Hint: A **blue fairy** gives you the identities

$$2\cos(x)\cos(nx) = \cos((n+1)x) + \cos((n-1)x)$$

$$2\cos(x)\sin(nx) = \sin((n+1)x) + \sin((n-1)x)$$

b) (4 points) Find the value of the sum of the squares

of all the Fourier coefficients of f.

Problem 13) (10 points)

a) (5 points) Solve the **modified heat equation**

$$u_t = 7u_{xx} + u_{xxxxxx}$$

with initial condition $u(x, 0) = \sin(11x) + 5\sin(22x)$.

b) (5 points) Solve the **modified wave equation**

$$u_{tt} = 7u_{xx} + u_{xxxxxx}$$

with initial condition $u(x, 0) = e^x$ and $u_t(x, 0) = 0$.

Hint: The **oracle of Delphi** gives you the identities

$$\int_0^\pi e^x \sin(nx)\, dx = n(1 - e^\pi(-1)^n)/(1 + n^2)$$

$$\int_0^\pi e^x \cos(nx)\, dx = (e^\pi(-1)^n - 1)/(1 + n^2)$$

Problem 14) (10 points)

We love determinants. Can not get enough of them. Anyway, here is an other one for a bonus 14 problem (the actual exam always has 13 problems as you might have noticed (1) . . .). Find the determinant of

$$A = \begin{bmatrix}
9 & 1 & 1 & 1 & 1 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
1 & 9 & 1 & 1 & 1 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
1 & 1 & 9 & 1 & 1 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
1 & 1 & 1 & 9 & 1 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
1 & 1 & 1 & 1 & 9 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 8 & 8 & 8 & 8 & 8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 8 & 8 & 8 & 8 & 8 \\
0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 8 & 8 & 8 & 8 & 8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 8 & 8 & 8 & 8 & 8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 8 & 8 & 8 & 8 & 8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 1 & 1
\end{bmatrix}.$$

(1) Actually, some students in the past have been even done statistics (big data!) like on what the expected number of True answers are in the TF-section of Math21b. As in average in the last dozen practice tests, there used to be 11.8 True from 20 problems, a strategy was to just mark everything True, scoring at least 12 points in that section. But we might (for fun) once design a TF problem 1), where all 20 problems are False ... (and then maybe have one True), but still get that 11.8 average down a bit. Anyway, instead of "big data" it can be a better strategy to understand as many TF problems as possible.