MWF 9 Oliver Knill
MWF 10 Jeremy Hahn
MWF 10 Hunter Spink
MWF 11 Matt Demers
MWF 11 Yu-Wen Hsu
MWF 11 Ben Knudsen
MWF 11 Sander Kupers
MWF 12 Hakim Walker
TTH 10 Ana Balibanu
TTH 10 Morgan Opie
TTH 10 Rosalie Belanger-Rioux
TTH 11:30 Philip Engel
TTH 11:30 Alison Miller

- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) (20 points) True or False? No justifications are needed.

1) T F The inverse of the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is equal to $A/2$.

Solution:
Direct check. Multiply A with itself to see

2) T F The sum of two eigenvectors v_1, v_2 of a matrix A is an eigenvector of A.

Solution:
Only if the two eigenvectors belong to the same eigenvalue, this is true.

3) T F If S is the matrix which contains an eigenbasis of a $n \times n$ matrix A as columns, then $AS = DS$, where D is a diagonal matrix.

Solution:
We should have $S^{-1}AS = B$.

4) T F A 2×2 matrix A is diagonalizable if $A + A^T$ is diagonalizable.

Solution:
$A + A^T$ is always diagonalizable, but A does not need to be.

5) T F For the function $f(x) = |\sin(3x)| + |x|$, all Fourier coefficients of $\cos(nx)$ are zero.

Solution:
The function is even.

6) T F All continuous functions satisfying
$2\pi f(x) = f(\pi(x+1)) + f(\pi(x-1))$ form a linear space.

Solution:
This by the way is a functional equation which appears in so-called Bernoulli convolutions.

7) T **F** If the sum of all algebraic multiplicities of a 3×3 matrix A is equal to 3, then the matrix is diagonalizable.

Solution:
It would be true if algebraic would be replaced by geometric.

8) **T** F Every 2×2 matrix can be written as a product of a rotation, a diagonal matrix and a shear.

Solution:
This follows from the QR decomposition.

9) T **F** A discrete dynamical system for which all the eigenvalues are negative is asymptotically stable.

Solution:
It would be true for continuous dynamical systems.

10) T **F** A system of linear equations $A\vec{x} = \vec{b}$ which is consistent has exactly one solution.

Solution:
Consistent means that there is one solution. In general, there is either no, 1 or infinitely many solutions.

11) **T** F A rotation composed with a reflection is an orthogonal transformation.

Solution:
Both transformations are orthogonal. Their composition is orthogonal too. Both length and angles are preserved.

12) **T** F If $f(x,t) = \sum_{n=1}^{\infty} \frac{1}{n^3} e^{-n^2 t} \sin(nx)$ solves the heat equation, then $\langle f(x,0), \sin(3x) \rangle = 1/27$, where $\langle f, g \rangle = \frac{2}{\pi} \int_0^\pi f(x) g(x) \, dx$.

Solution:
Check the third coefficient of the series.

13) **T** F For a 5×6 matrix A the equation $Ax = b$ has either zero or infinitely many solutions.

Solution:
It can not have exactly one solution.

14) T **F**

If the Jacobian matrix at a equilibrium point (a, b) of a nonlinear system $x' = f(x, y)$, $y' = g(x, y)$ is orthogonal, then the point (a, b) is asymptotically stable.

Solution:
The eigenvalues are purely imaginary in this case and do not have negative real part.

15) T **F**

If a matrix A is diagonalizable, and $A = QR$ is the QR decomposition, then R is diagonalizable.

Solution:
Chose Q to be a rotation in the plane and R a shear in the plane. The eigenvalues of $A = QR$ are in all except some special cases different and A is diagonalizable.

16) T **F**

$||3\sin(5x) + 4\cos(20x)|| = 25$, where the length $||f||$ of a function f is taken with respect to the inner product $\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\, dx$.

Solution:
Parseval gives $3^2 + 4^2 = 5^2$ but the left hand side is the length, not the length squared.

17) T **F**

If the discrete dynamical system $\vec{x}(t+1) = A\vec{x}(t)$ is asymptotically stable, then also $\vec{x}(t+1) = (A^2 + A)\vec{x}(t)$ is asymptotically stable.

Solution:
The eigenvalues of $A^2 + A$ can become larger in absolute value than 1.

18) **T** F

If $dx/dt = Ax$ is a differential equation where A is a 2×2 matrix with trace 10, then $(0, 0)$ can not be asymptotically stable.

Solution:
The sum of the eigenvalues is 10. It can therefore not be that both eigenvalues have negative real part.

19) **T** F

If A is a symmetric 4×4 matrix satisfying $A^2 = A$, then the algebraic multiplicity is at least 2 for some eigenvalue of A.

Solution:
The eigenvalues can only be $0, 1, -1$ but we have 4 eigenvalues. Two must be the same.

20) T F If A is a reflection at the line $3x = 4y$ and B is a projection onto $x = y$ then $A + B$ is diagonalizable.

Solution:
The matrix $A + B$ is symmetric as a sum of symmetric matrices and therefore diagonalizable.

Problem 2) (10 points) no justifications needed

a) (6 points) In this problem, we deal with geometric linear transformations in the plane. Match the transformations with the trace and determinant values to the left. Each transformation matches exactly one of the cases.

trace	det	enter A-F
1	0	
0	-1	
0	1	
2	1	
4	4	
-2	1	

label	transformation
A	rotation by 90 degrees
B	projection onto a line
C	dilation by 2
D	reflection at a line
E	shear
F	rotation by 180 degrees

b) (4 points) Match the initial value problem with their solutions. Each function matches exactly one of the differential equations.

enter A-D	initial value problem
	$f''(t) = 4, f(0) = 4, f'(0) = 0$
	$f''(t) + 4f(t) = 4, f(0) = 2, f'(0) = 2$
	$f''(t) - 4f(t) = -4, f(0) = 3, f'(0) = 0$
	$f'(t) - 4f(t) = 4, f(0) = 4$

label	solution
A)	$f(t) = e^{2t} + e^{-2t} + 1$
B)	$f(t) = 1 + \cos(2t) + \sin(2t)$
C)	$f(t) = 5e^{4t} - 1$
D)	$f(t) = 4 + 2t^2$

Solution:
a) BDAECF
b) DBAC

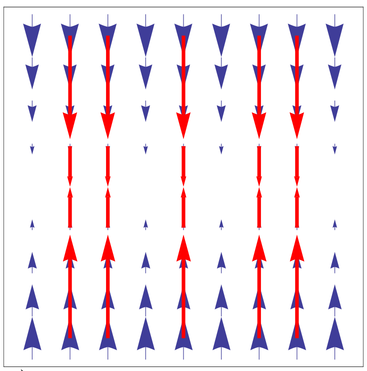
Problem 3) (10 points) no justifications needed

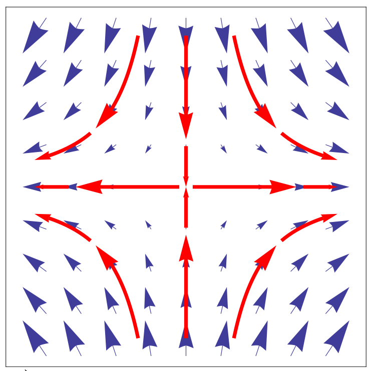
a) (5 points) Suppose you know that the eigenvalues of a 2×2 matrix A are $\lambda_1 = 2$ with eigenvector $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\lambda_2 = 3$ with eigenvector $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Check the boxes which apply.

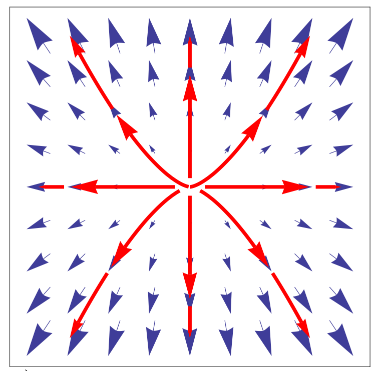
If $B =$	2 and 3 are eigenvalues of B	v_1 and v_2 are both eigenvectors of B
A^T		
$A^2 - A$		
A^{-1}		

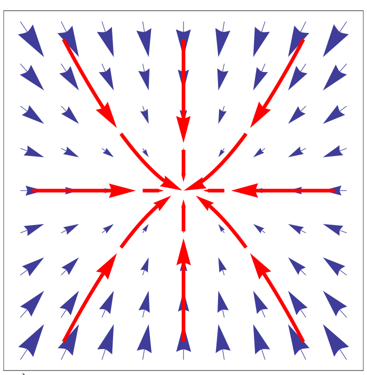
b) (5 points) Associate the dynamical system with the phase portraits.

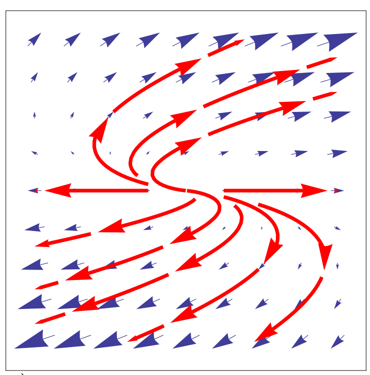
matrix	portrait a)-f) for $\frac{d}{dt}x = Ax$
$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$	
$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$	
$A = \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix}$	
$A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$	
$A = \begin{bmatrix} -1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$	

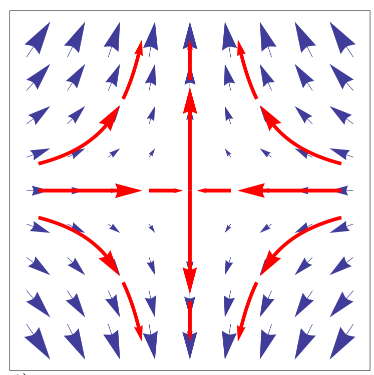

a)


b)


c)


d)


e)


f)

Solution:
a)

If $B =$	2 and 3 are eigenvalues of B	v_1 and v_2 are both eigenvectors of B
A^T	*	
$A^2 - A$		*
A^{-1}		*

b) $ECABF$.

Problem 4) (10 points)

Find all the solutions of the following system of linear equations. Use basic row reduction steps.

$$\left| \begin{array}{ccccccccccc} x & & & & & & & + & v & = 10 \\ & y & + & z & & & & & & = 6 \\ x & & & & + & u & & + & v & = 6 \end{array} \right|$$

Solution:
The augmented matrix is

$$B = \left| \begin{array}{ccccccc} 1 & 0 & 0 & 0 & 1 & | & 10 \\ 0 & 1 & 1 & 0 & 0 & | & 6 \\ 1 & 0 & 0 & 1 & 1 & | & 10 \end{array} \right| .$$

It row reduces to

$$\text{rref}(B) = \left| \begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & | & 10 \\ 0 & 1 & 1 & 0 & 0 & | & 6 \\ 0 & 0 & 0 & 1 & -1 & | & -4 \end{array} \right| .$$

There are two columns without leading 1 and therefore 2 free variables. Call them s, t and write

$$x = 10 - t, y = 6 - s, z = s, u = -4, v = t$$

to get

$$\begin{bmatrix} 10 \\ 6 \\ 0 \\ -4 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} .$$

Problem 5) (10 points)

Find the best function

$$f(x,y) = a\sin(x) + b\cos(y) = z$$

which fits the data points $(0,0,1), (\pi/2,0,2), (\pi/2,\pi/2,4)$ using the least square method.

Solution:
Write down the system of equation $A\vec{x} = \vec{b}$ as if all points were on the graph of $f(x,y)$:

$$\begin{aligned} 0 + b &= 1 \\ a + b &= 2 \\ a + 0 &= 4 . \end{aligned}$$

Here $\vec{x} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ and the system is

$$A\vec{x} = \vec{b} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} .$$

Now crank in the least square solution formula:

$$A^T \cdot A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} .$$

$$A^T\vec{b} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} .$$

to get the best solution vector

$$\vec{x}_* = (A^TA)^{-1}A^T\vec{b} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} .$$

The best function is therefore $\boxed{f(x,y) = 3\sin(x)}$.

Problem 6) (10 points)

a) (4 points) Find all the eigenvalues and eigenvectors of the matrix

$$B = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix} .$$

b) (1 point) is the continuous dynamical system $\frac{d}{dt}\vec{x}(t) = B\vec{x}(t)$ defined by the matrix B in a) asymptotically stable?

c) (4 points) A regular 3×3 transition matrix A defines a discrete dynamical system $\vec{x}(n+1) = A\vec{x}(n)$: which has the property that $A^n v$ converges to a multiple of an eigenvector belonging to the largest eigenvalue of

$$A = \begin{bmatrix} 1/3 & 0 & 2/3 \\ 0 & 2/3 & 1/3 \\ 2/3 & 0 & 1/3 \end{bmatrix} .$$

Find this eigenvector of A.

d) (1 point) Is the discrete dynamical system $\vec{x}(n+1) = A\vec{x}(n)$ from part c) asymptotically stable?

Solution:
a) This is a circular matrix problem. Since $B - 2I_4$ has eigenvalues $\mu_k = e^{2\pi ik/4}$ for $k = 0, 1, 2, 3$ and eigenvectors $v_k = [1, \lambda_k, \lambda_k^2, \lambda_k^3]^T$, the matrix B has eigenvalues $\lambda_k = 2 + e^{2\pi ik/4}$ with $k = 0, 1, 2, 3$ and the same eigenvectors. Written out, the eigenvalues are

$$\lambda_0 = 3, \lambda_1 = 2 + i, \lambda_2 = 1, \lambda_3 = 2 - i$$

with corresponding eigenvectors

$$\vec{v}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ -i \\ -1 \\ i \end{bmatrix} .$$

Alternatively, one could also compute the characteristic polynomial of B directly and get

$$f_B(\lambda) = (2 - \lambda)^4 - 1$$

and compute by hand the eigenvectors by row reduction.
b) It is not stable since not all eigenvalues have negative real part. The eigenvalue 3 is already a counter example.
c) The matrix has an eigenvalue 1 with eigenvector $[1, 1, 1]^T$. Since the trace is $4/3$ the sum of the other eigenvalues is $4/3 - 1 = 1/3$. The determinant, the product of the eigenvalues is $-2/9$. We can get from this the other eigenvalues $-1/3, 2/3$ which are in modulus smaller than 1. The largest eigenvalue is indeed 1.
d) The eigenvalue 1 is already positive so that there is no stability. Note that this is no contradiction to the property that a probability distribution converges to an equilibrium distribution. The question asks about the stability of the origin.

Problem 7) (10 points)

Assume the matrix A has an eigenbasis

$$\mathcal{B} = \{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \}$$

which belongs to the eigenvalues $1, 2$ and 3 given in the same order. Find the matrix A.

Solution:
The matrix which transforms us in the new basis is

$$S = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} .$$

In the new basis, the transformation is diagonal with the eigenvalues as diagonal entries

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} .$$

The matrix A is related to B with $B = S^{-1}AS$ so that $A = SBS^{-1}$. We invert S and perform the matrix multiplications

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & -2 & 3 \end{bmatrix} .$$

Problem 8) (10 points)

Let's have a look at the **multiplication table** we learned as kids in first grade:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 \\ 3 & 6 & 9 & 12 & 15 & 18 & 21 & 24 & 27 \\ 4 & 8 & 12 & 16 & 20 & 24 & 28 & 32 & 36 \\ 5 & 10 & 15 & 20 & 25 & 30 & 35 & 40 & 45 \\ 6 & 12 & 18 & 24 & 30 & 36 & 42 & 48 & 54 \\ 7 & 14 & 21 & 28 & 35 & 42 & 49 & 56 & 63 \\ 8 & 16 & 24 & 32 & 40 & 48 & 56 & 64 & 72 \\ 9 & 18 & 27 & 36 & 45 & 54 & 63 & 72 & 81 \end{bmatrix} .$$

Find all eigenvalues and eigenvectors of this symmetric matrix.

Solution:
The matrix A has rank 1 and so nullity 8 by the fundamental theorem of algebra. Since there is only one nonzero eigenvalue, it is equal to $\text{tr}(A) = 1 + 4 + 9 + ... + 81 = 285$ This eigenvalue has the eigenvector $[1, 2, 3, 4, 5, 6, 7, 8, 9]^T$, which also spans the image of A. The matrix is also symmetric and so a complete eigenbasis. To find the eigenspace to 0 we have to find the kernel of A which is the kernel of

$$\text{rref}(A) = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

We have

$$E_0 = \ker(A) = \ker(\text{rref}(A)) =$$

$$\text{span}\{ \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 9 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} \}$$

Problem 9) (10 points)

a) (2 points) Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 5 & 0 \\ 1 & 0 & 5 \end{bmatrix} .$$

b) (4 points) When studying the **"quintic threefold"** in string theory, the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 5 & 0 & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 & 0 & 0 \\ 1 & 0 & 0 & 5 & 0 & 0 \\ 1 & 0 & 0 & 0 & 5 & 0 \\ 1 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

appears. Find the determinant of A.

c) (4 points) **Werner Heisenberg** formulated quantum mechanics using matrices. The truncated $n \times n$ version of the position operator matrix is called A_n. We have for example

$$A_8 = \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{1} & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{4} & 0 & \sqrt{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{5} & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{6} & 0 & \sqrt{7} \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{7} & 0 \end{bmatrix}$$

Find $\det(A_2)$. Explain, why $\det(A_4) = (-3)\det(A_2)$ and $\det(A_6) = (-5)\det(A_4)$. Do the same for $\det(A_8) = (-7)\det(A_6)$. Use this to compute $\det(A_8)$.

Solution:
a) $\boxed{\det(A) = 15}$ by Laplace expansion.
b) $\boxed{\det(A) = 0}$ since the columns $v_1, ..., v_6$ of A satisfy

$$5v_1 - v_2 - v_3 - v_4 - v_5 - v_6 = 0 \; .$$

Remark: The matrix appears on page 910 in the article "Geometric Aspects of Mirror Symmetry, by D.R. Morrison, in Mathematics Unlimited, 2001 and Beyond, Springer.
Remark: Laplace expansion is not so effective here since it gives a bit too much work. Row reduction is the best way to proceed. An other possibility is to subtract 5 times the identity from A. The matrix $B = A - 5I$ has 4 eigenvalues 0 as well as eigenvalues 5 and -1. The eigenvalues of A are then $5-5, -1-5, 0-5, 0-5, 0-5, 0-5 = 0, -6, -5, -5, -5, -5$ and since the determinant is the product we get 0. This method would allow us to get determinants of $n \times n$ matrices of the same type as in a) and b).
c) $\det(A_2)_= - 1$. The asserted formulas follow by Laplace expansion (two steps each) and

$$\det(A_8) = (-1)(-3)(-5)(-7) = 105 \; .$$

Remark: Heisenberg, Born and Jordan introduced matrix mechanics in 1925. It was followed by Schroedingers wave mechanics. Because with a suitable basis on functions, one can describe any linear transformations with matrices, both descriptions of quantum mechanics are equivalent. Schroedingers was more successful.

Problem 10) (10 points)

Find the general solutions for the following differential equations:

a) (5 points)

$$f''(t) + f(t) = 3\sin(2t) + 1, f(0) = 3, f'(0) = 0$$

b) (5 points)

$$f''(t) - 2f'(t) + f(t) = t^2, f(0) = 2, f'(0) = -4$$

Solution:
a) The homogeneous equation is the harmonic osciallator. It has the solution $C_1 \cos(t) + C_2 \sin(t)$. To find a specific solution, we try $A \sin(t) + B$ and fix the constants A, B. The initial conditions fix the constants C_1, C_2¿

$$2\cos(t) + 2\sin(t) - \sin(2t) + 1$$

b) Since the left hand side is $(D-1)^2 f$ the solution to the homogeneous equation is $C_1 e^t + C_2 t e^t$. A specific solution is found by trying $At^2 + Bt + C$ and fixing the constants A, B, C. The constants C_1, C_2 finally are determined by the initial condition:

$$-4e^t - 4te^t + 4t + t^2 + 6 \ .$$

Problem 11) (10 points)

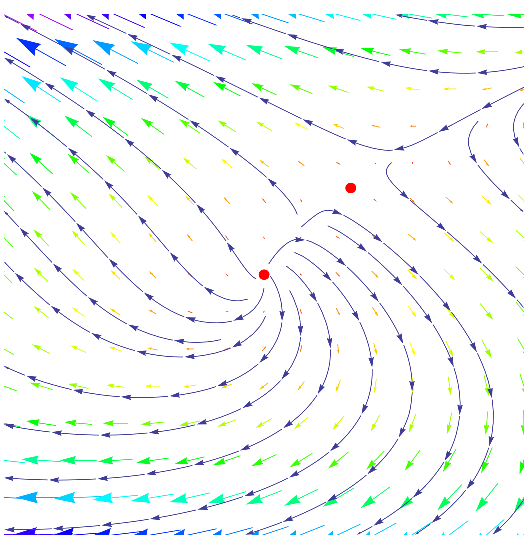
Analyze the solutions $(x(t), y(t))$ for the following nonlinear dynamical system

$$\begin{aligned} \frac{d}{dt}x &= x - y^2 \\ \frac{d}{dt}y &= y - x \end{aligned}$$

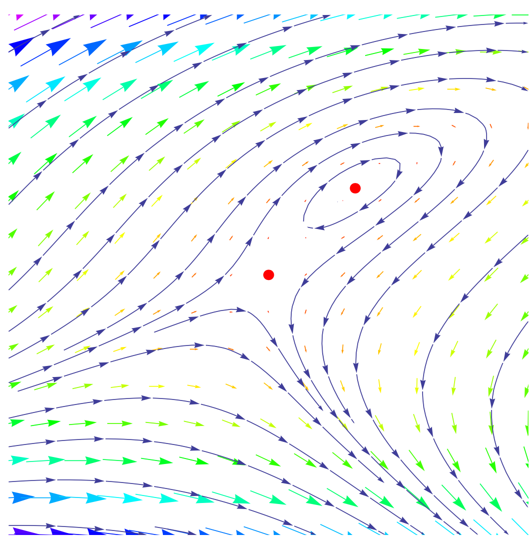
a) (3 points) Find the equations of the null-clines and find all the equilibrium points.

b) (4 points) Analyze the stability of all the equilibrium points.

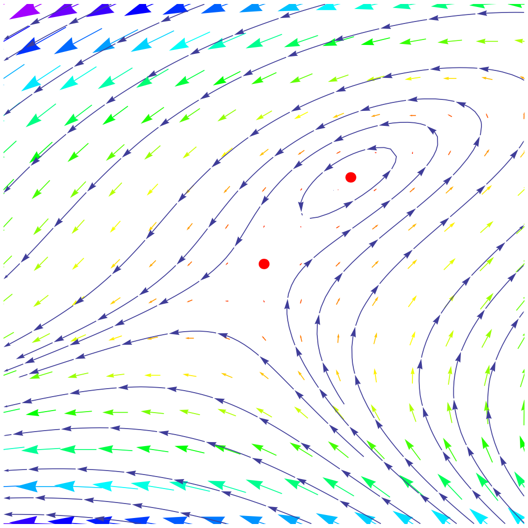
c) (3 points) Which of the phase portraits A,B,C,D below belongs to the above system?



A



B



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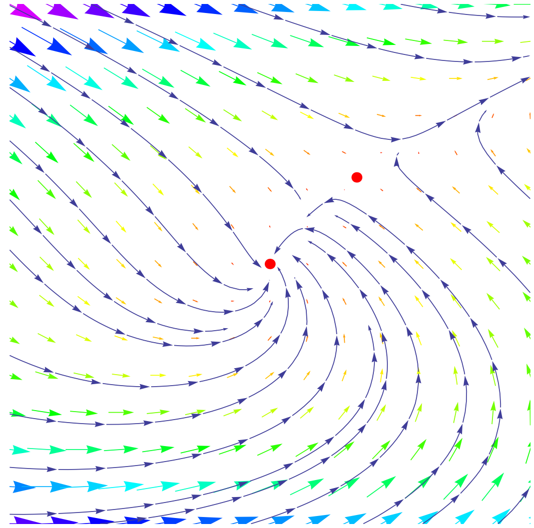


D

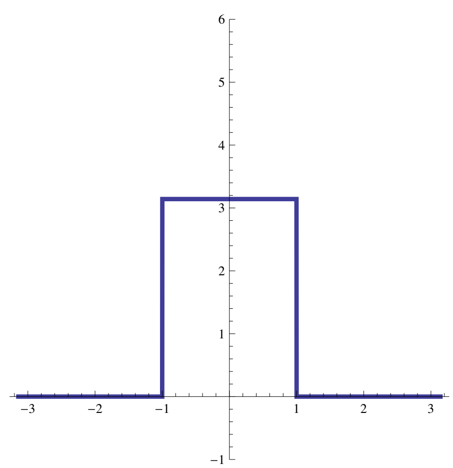
Solution:
a) The x-nullclines are $x = y^2$. The y-nullclines are $x = y$. They intersect in the points $(0,0)$ and $(1,1)$.
b) At $(0,0)$ the Jacobian matrix is $J = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ which has the eigenvalues $1, 1$. At $(1,1)$ the Jacobian matrix is $J = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}$ which has the eigenvalues $1 \pm \sqrt{2}$. One is positive and one is negative.
c) The correct picture is A).

Problem 12) (10 points)

a) (4 points) Find the Fourier series of the function

$$f(x) = \begin{cases} \pi & , \quad -1 \le x \le 1 \\ 0 & , \quad \text{else} \end{cases} .$$

The graph of the function is visible to the right.



Note: Leave terms like $\cos(n), \sin(n)$ as they are.

b) (3 points) Use Parseval's theorem to find the value of the sum

$$\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^2} .$$

c) (3 points) We add a parameter $0 \leq a \leq \pi$ in the above computation:

$$f(x) = \begin{cases} \pi & , \quad -a \leq x \leq a \\ 0 & , \quad \text{else} \end{cases} .$$

What is

$$g(a) = \sum_{n=1}^{\infty} \frac{\sin^2(na)}{n^2} \ ?$$

Solution:
a) The function f is even and has therefore a cos-series. We have $a_0 = \frac{2}{\pi} \int_0^1 \pi/\sqrt{2} \, dx = \sqrt{2}$ and

$$a_n = \frac{2}{\pi} \int_0^1 \pi \cos(nx) \, dx = \frac{2\sin(n)}{n} .$$

The Fourier series is

$$\sqrt{2}\frac{1}{\sqrt{2}} + \sum_{n=1}^{\infty} \frac{2\sin(n)}{n} \cos(nx) .$$

b) To use Parseval, first compute $||f||^2 = \frac{2}{\pi} \int_0^1 \pi^2 \, dx = 2\pi$ and $a_0^2 = 2$. We have

$$4 \sum_{n=1}^{\infty} \frac{\sin(n)^2}{n^2} = \sum_{n=1}^{\infty} a_n^2 = ||f||^2 - a_0^2 = 2\pi - 2$$

so that the sum is $\boxed{\frac{\pi-1}{2}}$.
c) The computation with the additional parameter is similar. We have $a_0 = \sqrt{2}a$ and $a_n = 2\sin(na)/a$ and $||f||^2 = \frac{2}{\pi} \int_{-a}^{a} \pi^2 \, dx = 2\pi a$. Now

$$4 \sum_{n=1}^{\infty} \frac{\sin(na)^2}{n^2} = ||f||^2 - a_0^2 = 2\pi a - 2a^2$$

and the sum is $\boxed{\frac{\pi a - a^2}{2}}$.
Remark. This is cool, since we can now also compute the value of the famous **cos-series**

$$\sum_{n=1}^{\infty} \frac{\cos(na)}{n^2}$$

as follows: the ouble angle formula $\cos(na) = 1 - 2\sin^2(na/2)$ shows

$$\sum_{n=1}^{\infty} \frac{\cos(na)}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - 2\sum_{n=1}^{\infty} \sin^2(na/2) = \frac{\pi^2}{6} - 2\frac{\pi a/2 - a^2/4}{2} = \frac{\pi^2}{6} - \frac{a\pi}{2} + \frac{a^2}{4} .$$

Problem 13) (10 points)

A **forest fire** temperature is given by a function $f(x,y,t)$ for (x,y) in a square $[0,\pi]\times[0,\pi]$. Since the burning intensity becomes bigger with more heat and additionally, the fire diffuses, the temperature $f(x,y,t)$ at the point (x,y) and time t satisfies the partial differential equation

$$f_t = T(f) = f_{xx} + f_{yy} + 11 \cdot f \ .$$

a) (2 points) The operator $T = D_x^2 + D_y^2 + 11$ has eigenfunctions $\sin(nx)\sin(my)$. What are the eigenvalues?

b) (4 points) Assume the fire has initially the temperature

$$f(x,y) = \sin(4x)\sin(7y) \ .$$

What is the temperature at a later time t? Does it die out?

c) (4 points) Now assume the fire has initially the temperature

$$f(x,y) = \sin(3x)\sin(y) \ .$$

What happens now?

Solution:
a) Apply T to the function $\sin(nx)\sin(my)$ to get $(-n^2 - m^2 + 11)\sin(nx)\sin(my)$. The eigenvalues are $\lambda_{n,m} = -n^2 - m^2 + 11$.
b) Since the initial function is an eigenfunction it satisfies the ordinary differential equation

$$f_t = \lambda_{4,7} f = (-4^2 - 7^2 + 11)f = -54f$$

we get

$$f(t,x,y)) = e^{-54t}\sin(4x)\sin(7y) \ .$$

This solution goes to zero as $t \to \infty$. The fire dies out.
c) The initial function is an eigenfunction to the eigenvalue 1. The initial function moves according to the ordinary differential equation

$$f_t = \lambda_{3,1} f = (-3^2 - 1^2 + 11)f = f \ .$$

which has the solution

$$f(t,x,y) = e^t \sin(3x)\sin(y) \ .$$

The fire grows. This story illustrates that in diffusion processes, high frequency modes damp away fast.

Problem 14) (10 points)

We have seen that the complex solutions to $\lambda^{30} - 1 = 0$ can be found by writing $1 = e^{2\pi i k}$ and taking the roots. Solve the folling 30'th order differential equation. We love the extreme.

$$D^{30} f - f = f^{''''''''''''''''''''''''''''''} - f = \sin(2t) \ .$$

Solution:
The homogeneous solution is solved by $C_1 e^{2\pi i t 1/30} + C_2 e^{e^{2\pi i t 2/30}} + ... C_{30} e^{e^{2\pi i t 30/30}}$. For a particular solution, plug in $A\sin(t)$ into the differential equation. We have $A 2^{30} \sin(2t) - A\sin(2t) = 1$ so that $A = 1/(2^{30} - 1)$. The general solution is the sum of the homogeneous and the special solution.