MWF 9 Oliver Knill
MWF 10 Jeremy Hahn
MWF 10 Hunter Spink
MWF 11 Matt Demers
MWF 11 Yu-Wen Hsu
MWF 11 Ben Knudsen
MWF 11 Sander Kupers
MWF 12 Hakim Walker
TTH 10 Ana Balibanu
TTH 10 Morgan Opie
TTH 10 Rosalie Belanger-Rioux
TTH 11:30 Philip Engel
TTH 11:30 Alison Miller

- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) (20 points) True or False? No justifications are needed.

1) T **F**

If a linear system $Ax = 0$ has at least one solution, then the system $Ax = b$ has at least one solution for all b.

Solution:
Consistent means that there is at least one a solution. There is the solution $x = 0$ but not necessarily for the inhomogeneous solution.

2) **T** F

If A is an orthogonal matrix, then all matrix entries A_{ij} satisfy $|A_{ij}| \leq 1$ for all i, j.

Solution:
This follows from the fact that the row vectors have length 1.

3) **T** F

The transformation $T(f)(x) = f(x^2) - 23f(x)$ is linear on the space of all polynomials.

Solution:
Check the three conditions.

4) **T** F

If a smooth function f on $[-\pi, \pi]$ has a sin-Fourier expansion then it satisfies $\int_{-\pi}^{\pi} f(x)\, dx = 0$.

Solution:
The function is perpendicular to every even function, especially to the constant function.

5) T **F**

The characteristic polynomials of two real $n \times n$ matrices A, B satisfy $f_A(\lambda) + f_B(\lambda) = f_{A+B}(\lambda)$.

Solution:
The formula does not even hold in one dimensions.

6) T **F**

The function $f(t) = 23e^{10t}$ is an eigenfunction with eigenvalue 23 of the linear operator $T = D$, where $Df = f'$ is the differentiation operator on $C^\infty(\mathbf{R})$.

Solution:
It is an eigenfunction to the eigenvalue 10.

7) T F The matrix $(A^{23})(A^{23})^T$ is diagonalizable, if A is a real $n \times n$ matrix.

Solution:
Yes, the matrix is symmetric.

8) T F The initial value problem $f''(x) + 23f'(x) + 10f(x) = x + e^x$, $f'''(0) = 0$ has exactly one solution.

Solution:
Without initial condition, the solution space is two dimensional. Fixing $f'''(0)$ determines one constant. The solution space is one dimensional. There are many solutions

9) T F The transformation $T(f)(x) = \sin(x)f(\sin(x))$ is a linear transformation on the space $X = C^\infty(\mathbf{R})$ of smooth functions on the real line.

Solution:
We check three conditions: $T(0) = 0, T(f + g) = T(f) + T(g), T(\lambda f) = \lambda T(f)$ as well as the condition that T maps the space into itself.

10) T F The set X of smooth functions $f(x, t)$ of two variables which satisfy the partial differential equation $f_{ttt} - f_{xxx} = f_x$ is a linear space.

Solution:
Yes, if we add two functions which satisfy this differential equation, then the sum also satisfies this differential equation. Also a scaled function satisfies the differential equation. And the zero function also satisfies this differential equation.

11) T F If A is 23×23 matrix of rank 23, then it has an eigenvalue 0.

Solution:
Having an eigenvalue 0 is equivalent to have a nontrivial kernel.

12) T F The vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ has the $\mathcal{B} = \{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}\}$-coordinates $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Solution:
Yes, $[1, 0] = [1, 2] - [0, 2]$.

13) T F

If all the geometric multiplicities of the eigenvalues of a matrix are equal to the algebraic multiplicities, then the matrix is diagonalizable.

Solution:
We have an eigenbasis.

14) T F

For a differential equation $\frac{d}{dt}x = f(x,y), \frac{d}{dt}y = g(x,y)$, every equilibrium point is an intersection of two nullclines.

Solution:
If a x-nullclines intersects a y-nullcline consists of equilibrium points.

15) T F

If $z = 2i$ then $\sqrt{z} = 1 + i$ or $1 - i$.

Solution:
The polar angle of z is divided by 2 when we take the square root. The correct answer is $1 + i$ or $-(1 + i)$.

16) T F

The determinant and trace of a 2×2 matrix A always satisfy the inequality $\text{tr}(A) \leq \det(A)$.

Solution:
It is already not true for diagonal matrices.

17) T F

The QR decomposition of an upper triangular matrix A with positive diagonal entries is $A = QR$, where $R = A$ and $Q = 1_n$.

Solution:
It is already the QR decomposition.

18) T F

If the trace and the determinant of a 2×2 matrix A are both zero, then A is the zero matrix.

Solution:
The matrix $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ has trace 0 and determinant 0.

19) T F

The discrete dynamical system $x(t+1) = x(t) + 23\,x(t-1)$ has the property that $|x(t)| \to \infty$ for all nonzero initial conditions $(x(0), x(1))$.

Solution:
This is a variant of the Fibonacci system

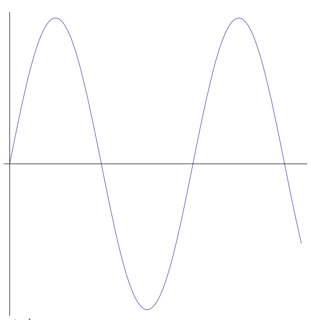
20) T F $|| \sin(x) + \cos(23x)|| = \sqrt{2}$, where $||f|| = \sqrt{\langle f, f\rangle}$ is the length of the function f with respect to the inner product $\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx$.

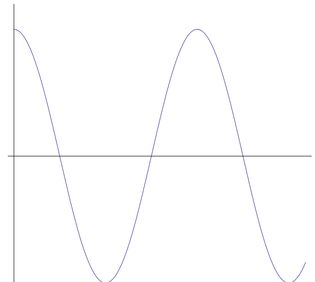
Solution:
This is a direct consequence of the Perceval identity.

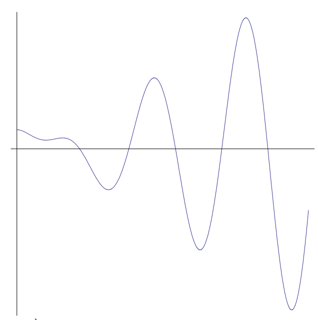
Problem 2) (10 points)

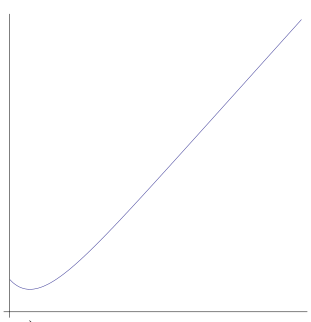
No justifications are needed in this problem. Match the equations with the solution graphs $f(x)$. Note that the graphs are not necessarily to scale. Enter five of the six choices A,B,C,D,E,F here:

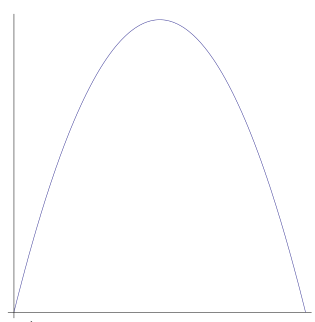
	$f''(t) + f(t) = \sin(t), f(0) = 1, f'(0) = 0$
	$f'(t) = -f(t), f(0) = 1$
	$f''(t) = -1, f(0) = 0, f'(0) = 1$
	$f''(t) = -\sin(t), f(0) = 0, f'(0) = 1$
	$f'(t) + f(t) = t, f(0) = 1$

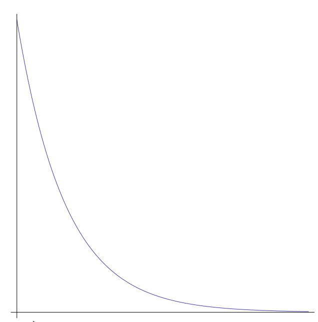

A)


B)


C)


D)


E)


F)

Solution:
The first equation is a driven harmonic oscillator in resonance, the second equation describes exponential decay, the third is a free fall problem, the fourth is solved by integrating twice, the last is exponential decay driven by a linear term which makes the solution approach the linear solution. The solution formula is C,F,E,A,D.

Problem 3) (10 points)

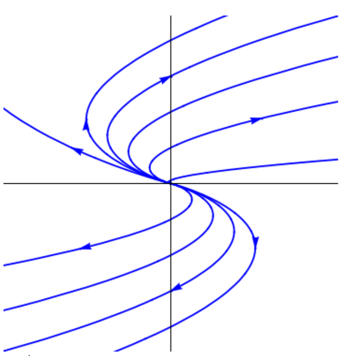
No justifications are needed in this problem. Check all boxes which apply except in the first column where you have to enter your choices of A)-F). We abbreviate the term "asymptotically stable" with "stable" and "diagonalizable" means diagonalizable over the complex numbers.

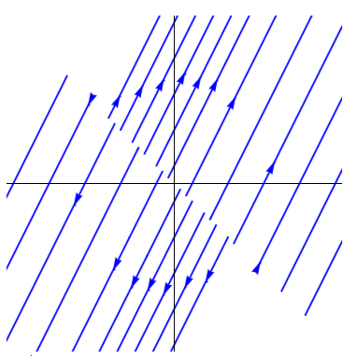
matrix A	phase A)-F)	$\frac{d}{dt}x = Ax$ stable	$x(t+1) = Ax(t)$ stable	A diagonalizable
$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$				
$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$				
$\begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}$				
$\begin{bmatrix} 0 & 2 \\ -2 & -1 \end{bmatrix}$				
$\begin{bmatrix} 3/4 & 4 \\ 0 & 2/4 \end{bmatrix}$				

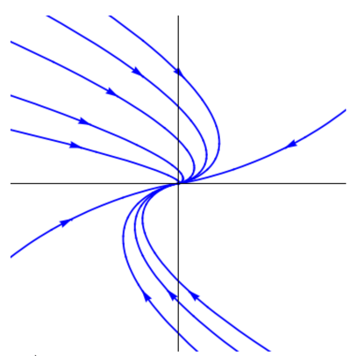
Below are 6 portraits for the continuous dynamical system

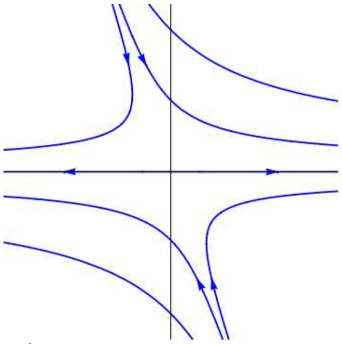
$$\frac{d}{dt}x = Ax .$$

Please enter 5 of the 6 letters A-F in the table above.

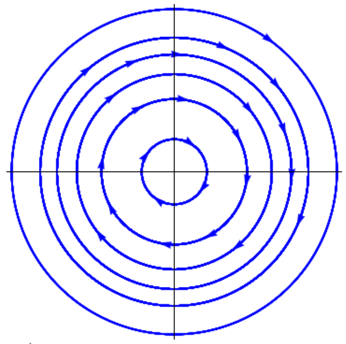

A)


B)

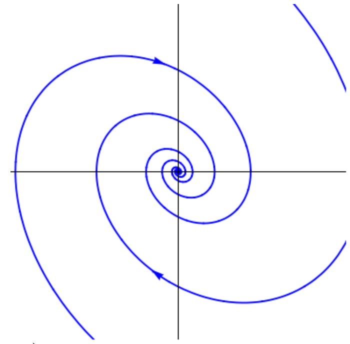

C)



D)



E)



F)

Solution:
B,C,D,F,A. The second and fourth are stable for the continuous system. The last is stable for the discrete system. Only the second is not diagonalizable.

Problem 4) (10 points)

Find all the solutions of the system of linear equations for the variables x, y, z, u.

$$\left| \begin{array}{ccccccc} x & - & y & & & & = 4 \\ & & y & - & z & & = 5 \\ & & & & z & + & u = 7 \end{array} \right|$$

Solution:
We row reduce the augmented matrix $B = [A|b]$ to get

$$\text{rref}(B) = \begin{bmatrix} 1 & 0 & 0 & 1 & | & 16 \\ 0 & 1 & 0 & 1 & | & 12 \\ 0 & 0 & 1 & 1 & | & 7 \end{bmatrix}$$

There are three leading ones and one column with a free variable (u). The solution is $x = 16 - u, y = 12 - u, z = 7 - u, u = u$.

$$\begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 16 \\ 12 \\ 7 \\ 0 \end{bmatrix} - u \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} .$$

Problem 5) (10 points)

Find all the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 23001 & 3 & 5 & 7 & 9 & 11 \\ 1 & 23003 & 5 & 7 & 9 & 11 \\ 1 & 3 & 23005 & 7 & 9 & 11 \\ 1 & 3 & 5 & 23007 & 9 & 11 \\ 1 & 3 & 5 & 7 & 23009 & 11 \\ 1 & 3 & 5 & 7 & 9 & 23011 \end{bmatrix} .$$

As usual, document all your reasoning.

Solution:
The eigenvalues of $B = A - 23000$ are 0 (with algebraic multiplicity 5) and $1+3+5+7+9+11 = 36$ with algebraic multiplicity 1.
Five of the eigenvectors span the kernel of B and are

$$\begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 9 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 11 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} .$$

The eigenvector to the eigenvalue 36 is

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} .$$

The matrix A has the same eigenvectors than A and the eigenvalues 23000 with multiplicity 5 and eigenvalue 23036 with multiplicity 1.

Problem 6) (10 points)

a) (3 points) Find the 3×3 matrix A belongs to the linear transformation which reflects a vector at the x axes.

b) (4 points) Find the 3×3 matrix B belongs to the linear transformation which projects onto the axes $x = y = z$.

c) (3 points) Find the matrix

$$AB - BA ,$$

the so called **commutator** of A and B.

Solution:
a) The matrix is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

One gets the matrix by constructing the columns as images of the basis vectors.
b) Again, to get the matrix, we find the images of the basis vectors:

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} /3 \; .$$

c) The commutator is

$$AB - BA = \begin{bmatrix} 0 & 2 & 2 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} /3$$

Problem 7) (10 points)

Find the function $z = axy + bx^2$ which best fits the data

x	y	z
1	1	4
1	2	6
1	0	8

Solution:
We want to find the least square solution of

$$\begin{aligned} a+b &= 4 \\ 2a+b &= 6 \\ b &= 8 \end{aligned}$$

It is of the form $A \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix}$ matrix

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 0 & 1 \end{bmatrix} .$$

We compute $A^T A = \begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix}$. $A^T \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \end{bmatrix}$. The solution is

$$\begin{bmatrix} a \\ b \end{bmatrix} = (A^T A)^{-1} A^T \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} .$$

With our least square solution $\boxed{a = -1, b = 7}$, the best function is $\boxed{-xy + 7x^2}$.

Problem 8) (10 points)

Solve the difference equation

$$x_{n+1} - x_n = 5y_n$$
$$y_{n+1} - y_n = 5x_n$$

with initial condition $x_0 = 5, y_0 = 7$.

Solution:
The eigenvectors are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to the eigenvalue 6 and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ to the eigenvalue -4. The initial condition is $6v_1 - v_2$. The closed form solution is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = 6 \cdot 6^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-1) \cdot (-4)^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} .$$

Problem 9) (10 points)

Find the determinants of the following matrices:

a)

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \end{bmatrix}$$

b)

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 5 & 0 & 0 \end{bmatrix}$$

c)

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 1 & 1 & 6 \end{bmatrix}$$

Solution:
a) After row reduction we have identical rows. Each row reduction step multiplies the determinant with a nonzero constant. Because the row reduced determinant is 0, the determinant of the original matrix is $\boxed{0}$.
b) Laplace expansion and keeping track of the signs gives $\boxed{-120}$.
P.S. Alternatively, we can use the permutation definition of the determinant. We see that the nonzero entries in the matrix form a permutation pattern. The determinant is the product of the elements multiplied by the signature which is -1 in this case.
c) This is a partitioned matrix with two upper and lower triangular matrices. The determinant is the product of the determinants of the submatrices which is the product of the diagonal elements: $\boxed{540}$;

Problem 10) (10 points)

Find the general solutions for the following differential equations:

a) (3 points) $f''(t) = t^2 + 3$.

b) (3 points) $f''(t) + f(t) = t^2 + 3$

c) (4 points) $f''(t) + 6f'(t) + 9f(t) = 2e^{-3t}$

Solution:
a) $3t^2/2 + t^4/12 + C_1 + tC_2$.
b) $1 + t^2 + C_1 \cos(t) + C_2 \sin(t)$. This is the harmonic oscillator.
c) $t^2 e^{-3t} + C_1 e^{-3t} + tC_2 e^{-3t}$. This is a situation where both e^{-3t} as well as te^{-3t} were homogeneous solutions so that one has to try with $At^2 e^{-3t}$.

Problem 11) (10 points)

We analyze the nonlinear dynamical system

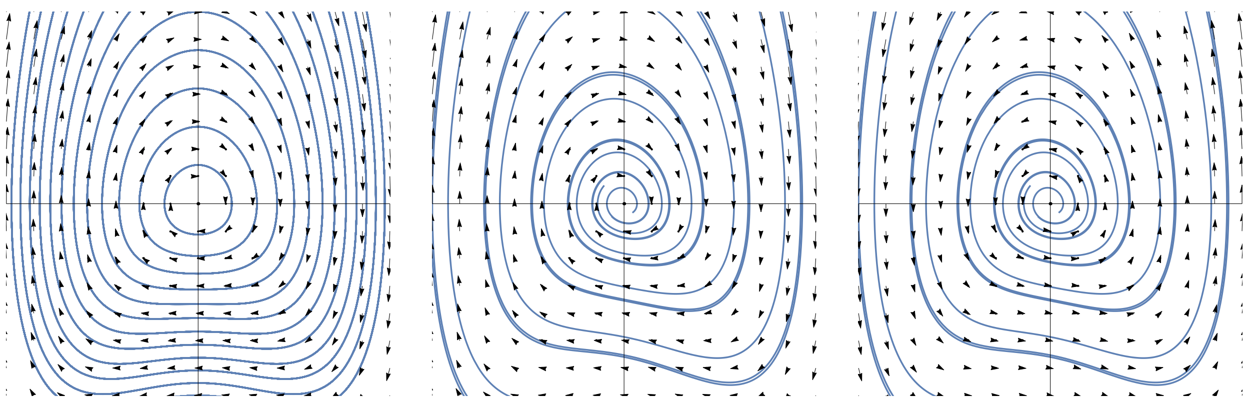
$$\begin{aligned} \frac{d}{dt}x &= y \\ \frac{d}{dt}y &= -x - xy - 2x^3 \end{aligned}$$

It is a variant of a **van der Pool oscillator**.

a) (3 points) Find the equations of the nullclines and find all the equilibrium points.

b) (4 points) Analyze the stability of all the equilibrium points.

c) (3 points) Which of the phase portraits A,B,C below belongs to the above system?



A B C

Solution:
a) The nullclines are the line $y = 0$ as well as the curve $x = 0$ and the parabola $y = -1 - 2x^2$.
The equilibrium points are the intersection of the nullclines which is the point $(0, 0)$.
b) The Jacobian matrix is $J = \begin{bmatrix} 0 & 1 \\ -1 - 6x^2 - y & -x \end{bmatrix}$. At $(0, 0)$ it is $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ which has eigenvalues $i, -i$. The origin is an elliptic fixed point asymptotically resembling the harmonic oscillator.
c) The phase spaces B and C have an equilibrium point with complex eigenvalues with a real part, either positive or negative. It is phase space $\boxed{A}$.

Problem 12) (10 points)

a) (5 points) Find the Fourier series of the function $f(x) = \begin{bmatrix} 1 & , |x| < \pi/4 \\ 0 & , |x| \geq \pi/4 \end{bmatrix}$. You do not have to simplify terms which look like $\sin(n\pi/3)$ or similar.

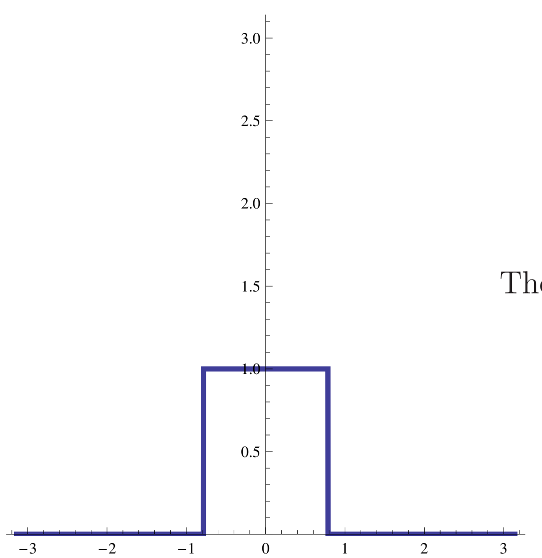
b) (5 points) The function $g(x) = x^3 - \pi^2 x$ has the Fourier series

$$g(x) = \sum_{n=1}^{\infty} \frac{12(-1)^n}{n^3} \sin(nx) .$$

What is

$$\sum_{n=1}^{\infty} \frac{1}{n^6} ?$$

This number is called $\zeta(6)$, the value of the **Riemann Zeta function** at 6.



The function $f(x)$ in problem 12a).

Solution:
a) The function is even. It has a cos series. We compute $a_0 = (2/\pi)\pi/(4\sqrt{2}) = \frac{\sqrt{2}}{4}$ and $a_n = \frac{2}{n\pi}\sin(n\pi/4)$. The Fourier series is

$$f(x) = \frac{\sqrt{2}}{4}\frac{1}{\sqrt{2}} + \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin(n\pi/4)\cos(nx) \; .$$

b) Perceval's theorem $||g||^2 = \sum_{n=1}^{\infty} b_n^2$ shows that the result $144\zeta(6) = \sum_n b_n^2$ is

$$\frac{2}{\pi}\int_0^{\pi} (x^3 - \pi^2 x)^2 \; dx = \frac{16\pi^6}{105} \; .$$

From the Parseval identity we get

$$\zeta(6) = \frac{1}{144}\sum_n b_n^2 = \frac{1}{144}\frac{16\pi^6}{105} \; .$$

Therefore $\boxed{\zeta(6) = \pi^6/945}$. P.S. We have in the handout computed $\zeta(2)$, in the practice exam and review $\zeta(4)$ and now in the final $\zeta(6)$. You can convince yourself that in the same way we can compute $\zeta(n)$ for every even n. Unfortunately, we can not proceed like this to compute $\zeta(n)$ for other integers like $n = 3$. The function $\zeta(z)$ is believed to have all zeros on the line Re(z) = 1/2. This is the famous Riemann hypothesis, the probably most important unsolved problem in mathematics.

Problem 13) (10 points)

Picture: Sean Oughton, Department of Mathematics, University of Waikato.

The partial differential equation

$$u_t = u_{xx} + bu = (D^2 + b)u$$

is a model for a **reaction diffusion process**, where a thermal process produces additional heat bu proportional to the given heat u. It could model rubbing a match at a matchbox.

a) (3 points) Show that $\sin(nx)$ is an eigenvector=eigenfunction of the operator $T = D^2 + b$ in the PDE $u_t = T(u)$. What is the eigenvalue?

b) (2 points) Solve the system for initial condition $u(x,0) = \sin(3x) + 2\sin(5x)$ in the case $b = 1$.

c) (2 points) Solve the system for initial condition $u(x,0) = \sin(3x) + 2\sin(5x)$ and general b.

d) (3 points) For large enough b, the heat production overcomes the dissipation. Assuming still the initial condition $u(x,0) = \sin(3x) + 2\sin(5x)$, find the threshold value b_0 so that for $b > b_0$, the temperature $u(x,t)$ grows indefinitely and the match lights up.

Solution:
a) $(D^2 + b)\sin(nx) = (-n^2 + b)\sin(nx)$. The eigenvalue is $\boxed{-n^2 + b}$.
b) The solution for $b = 1$ is: $\boxed{\sin(3x)e^{(-3^2+1)t} + 2\sin(5x)e^{(-5^2+1)t}}$.
c) The solution for general b is $\boxed{\sin(3x)e^{(-3^2+b)t} + 2\sin(5x)e^{(-5^2+b)t}}$
d) If $\boxed{b > 9}$, then the first part of the solution $\sin(3x)e^{(-3^2+b)t}$ grows to infinity. If $b > 25$, as the second part of the solution $2\sin(5x)e^{(-5^2+b)t}$ goes to infinity, but these two parts can never cancel because they are perpendicular to each other in the space $C^\infty([0,\pi])$.
P.S. You see that lower frequencies become unstable much faster than high frequency. If our initial condition would have been $\sin(100x)$, the solution would become unstable only for $b > 10'000$.

Problem 14) (10 points)

A symmetric $n \times n$ matrix A has the QR decomposition $A = QR$.

a) (2 points) Verify that the matrix $B = RQ$ has the same eigenvalues than A.

b) (2 points) Is the map T which assigns to A the matrix B a linear map?

c) (2 points) What happens if T is applied to a diagonal matrix?

d) (4 points) An important method to compute the eigenvalues of A is to iterate the map T. This leads to matrices $A_1, A_2, \ldots$ which converge to a diagonal matrix, a fact which you can take for granted here. Demonstrate that this process also produces an orthonormal set of eigenvectors of A.

This QR method to find eigenvalues and eigenvectors has been discovered independently by the mathematicians **J.G.F. Francis** and **Vera N. Kublonovskaya** in 1961. Its a simple method compaed to finding roots of characteristic polynomials and finding kernels for each root.

Solution:
a) $B = RQ = Q^{-1}QRQ = Q^{-1}AQ$ is similar to A (if you write $Q = S$, then it becomes more obvious $B = S^{-1}AS$ is the definition of similiarity). The matrix B has therefore the same eigenvalues than A.
b) No, the map is not linear. While it satisfies $T(0) = 0$, where 0 is the zero matrix and $T(\lambda A) = \lambda T(A)$, the property $T(A + B) = T(A) + T(B)$ is not true for almost all choices. An example is $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ with $T(A) = A$ and $B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ with $T(B) = \begin{bmatrix} 6 & 13 \\ -3 & 6 \end{bmatrix}/5$. c) Diagonal matrices are fixed $T(A) = A$ if A is diagonal.
d) $A_1 = Q_1^{-1}AQ, A_2 = Q_2^{-1}A_1Q = Q_2^{-1}Q_1^{-1}AQ_1Q_2$, ... etc shows that the limit of orthogonal matrices $S = \lim_{n\to\infty} Q_1...Q_n$ contains the orthonormal eigenbasis as columns. Here is the producedure. We start with A and end up with a matrix B as well as with a conjugation matrix A (which we usually call S)

$$\begin{aligned} A_1 &= Q_1^T AQ1 \\ A_2 &= Q_2^T A_1 Q_2 = Q_2^T Q_1^T AQ_1Q_2 \\ A_3 &= Q_3^T A_2 Q_3 = Q_3^T Q_2^T Q_1^T AQ_1Q_2Q_3 \\ \text{converges to } & B = Q^T AQ \end{aligned}$$

The columns of the limiting $Q = \lim_{n\to\infty} Q_1Q_2 \cdots Q_n$ contain the eigenvectors of A. Pretty cool. Remember that for a large matrix, we would have to compute the characteristic polynomial, find the roots, then the kernel of each root to get the eigenvectors and possibly make a Gram-Schmidt process to find an orthonormal eigenbasis. The QR-algorithm described here does everything automatically. We end up with an orthonormal eigenbasis and eigenvalues using a simple iterative process.