Homework 4: Linear transformations

This homework is due on Monday, February 5, respectively on Tuesday February 6, 2018.

1 Which of the following transformations are linear? If it is, find the matrix A which implements the transformation.

a) $T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3y + x \\ 0 \\ x^2 - y \end{bmatrix}$ b) $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7^2 y \\ -3x \\ x \end{bmatrix}$ c) $T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x + 2y \\ y + z \\ 0 \end{bmatrix}$

d) $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2e^y \\ x \end{bmatrix}$ e) $T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z + x \\ -3y \end{bmatrix}$ f) $T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

2 Find the inverse of the following linear transformations $x \mapsto Ax$ or state that it is not invertible

a) $A = \begin{bmatrix} 4 & 1 \\ 8 & 2 \end{bmatrix}$ b) $A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$ c) $A = \begin{bmatrix} 0 & 1 \\ 0 & 4 \end{bmatrix}$

d) $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$, e) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, f) $A = \begin{bmatrix} 3 & -2 \\ -6 & 4 \end{bmatrix}$

g) Verify that $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} / (ad - bc)$ is the inverse of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

if $ad \neq bc$.

We will learn how to invert a matrix later. For now, get the inverse by solving $Ax = e_k$, rendering the k'th column of A^{-1}.
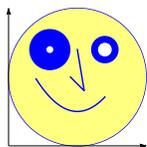
3 For each of the matrices, sketch the effect of the linear transformation $T(x) = Ax$ on the face.

a) $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ c) $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ e) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ f) $\begin{bmatrix} 1/3 & 0 \\ 0 & 2 \end{bmatrix}$

4 a) Let $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$. Which matrix A implements the transformation

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \to v \times x = \begin{bmatrix} v_2 x_3 - v_3 x_2 \\ v_3 x_1 - v_1 x_3 \\ v_1 x_2 - v_2 x_1 \end{bmatrix}.$$

b) Which matrix A implements the transformation

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \to v \cdot x = [v_1 x_1 + v_2 x_2 + v_3 x_3] ?$$

c) Is there a matrix which implements the transformation $(x, y) \to (x + 1, y + 2)$? If yes, write it down.

5 Find the linear transformation which reflects at the z-axes, then rotates by 180 degrees around the x-axes, then reflects at the xy-plane. Draw the images of thre three basis vectors $e_1 = [1, 0, 0]^T$, $e_2 = [0, 1, 0]^T$ and $e_3 = [0, 0, 1]^T$ to build the matrix. (To save space v^T is a column vector if v is a row vector).

Main properties

A map T mapping x to $T(x)$ is a **linear transformation** if there is a matrix A such that $T(x) = Ax$. The transformation is invertible if x can be obtained uniquely from b. In that case the inverse is again a linear transformation.

The columns of the matrix play a key role. The image of the vector e_1 is the first column, the image of e_2 the second column etc. When solving $Ax = e_k$ we get the k'th column of the inverse.