

## Homework 17: Determinants II

This homework is due on Monday, March 19, respectively on Tuesday, March 20, 2018. Its a good idea to finish this before spring break!

- 1 a) We find here the determinant of the  $5 \times 5$  matrix  $A$  for which the entry  $A_{km}$  is  $\phi(k + m)$ , where  $\phi$  is the Euler totient function giving the number of positive integers less than  $n$  that are co-prime to  $n$ . Use row reduction to find the determinant:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 & 1 \\ 1 & 1 & 3 & 1 & 1 \\ 1 & 2 & 1 & 4 & 1 \\ 1 & 1 & 1 & 1 & 5 \end{bmatrix}.$$

- b) Find the determinant of the GCD matrix of size  $500 \times 500$ . The result has only 1009 digits. What is the first digit? Excessive homework? No! You have all spring break ...

Hint: use a machine:

`M=500; Det[Table[GCD[n,k],{n,M},{k,M}]];`

- 2 a) Find the determinant of

$$B = \begin{bmatrix} 4 & 6 & 0 & 0 & 0 & 0 \\ 4 & 3 & 3 & 0 & 0 & 0 \\ 4 & 4 & 4 & 3 & 0 & 0 \\ 4 & 4 & 4 & 4 & 3 & 0 \\ 4 & 4 & 4 & 4 & 4 & 3 \\ 4 & 4 & 4 & 4 & 4 & 4 \end{bmatrix}.$$

- b) Find the determinant of  $9B$ .

- 3 a) Find the determinant of

$$A = \begin{bmatrix} 3 & 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 & 2 & 2 \\ 0 & 6 & 3 & 2 & 2 & 2 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & 2 & 4 & 4 \end{bmatrix}$$

- b) Find the determinant of  $A^5$ .

4 Argue geometrically why the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

has maximal absolute determinant  $|\det(A)|$  among all matrices with entries in  $\{-1, 1\}$ .

5 a) Find  $A, B$  such that  $\det(A - B) \neq \det(A) - \det(B)$ .

b) What values can an orthogonal matrix have?

c) Verify that  $|\det(A)|$  only depends on  $R$  if  $A = QR$  is the QR factorization.

## Determinants II

Determinants can be computed using row reduction: If during row reduction  $m$  swapping operations have occurred and the scaling factors are  $c_1, \dots, c_k$ , then

$$\det(A) = \frac{(-1)^m}{c_1 \cdots c_k} \det(\text{rref}(A))$$

Here are some more properties:

- $|\det(A)|$  is the volume of a parallel epiped
- $\det(AB) = \det(A)\det(B)$
- $\det(A^T) = \det(A)$
- $\det(A^n) = (\det(A))^n$
- $\det(A^{-1}) = 1/\det(A)$