Homework 22: Stability

This homework is due on Friday, March 30, respectively on Tuesday, April 3, 2018.

1 Determine the stability of the dynamical system $x(t+1) = Ax(t)$:

a) $\begin{bmatrix} 0.2 & 0.3 & 0.4 \\ 0.2 & 0.3 & 0.4 \\ 0.2 & 0.3 & 0.4 \end{bmatrix}$.

b) $\begin{bmatrix} 0.9 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -2 & 1 \end{bmatrix}$.

2 For which constants a is the system $x(t+1) = Ax(t)$ stable?

a) $A = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}$.

b) $A = \begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix}$.   c) $A = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$.

3 For which real values k does the drawing rule

$$x(t+1) = x(t) - ky(t)$$
$$y(t+1) = y(t) + kx(t+1)$$

produce trajectories which are ellipses? Write the system first as a discrete dynamical system $v(t+1) = Av(t)$ and look for the k for which the eigenvalues λ_k satisfy $|\lambda_k| = 1$.

4 Find the eigenvalues of

$$A = \begin{bmatrix} 0 & a & b & c & 0 & 0 \\ 0 & 0 & a & b & c & 0 \\ 0 & 0 & 0 & a & b & c \\ c & 0 & 0 & 0 & a & b \\ b & c & 0 & 0 & 0 & a \\ a & b & c & 0 & 0 & 0 \end{bmatrix}$$

Where a, b and c are arbitrary constants. Verify that the discrete dynamical system is stable for $|a| + |b| + |c| < 1$.

5 In the following, answer each question with a short explanation. We say A is stable if the origin $\vec{0}$ is a stable equilibrium.
a) True or false: the identity matrix is stable.
b) True or false: the zero matrix is stable.
c) True or false: every horizontal shear is stable.
d) True or false: any reflection matrix is stable.
e) True or false: A is stable if and only if A^T is stable.
f) True or false: A is stable if and only if A^{-1} is stable.
g) True or false: A is stable if and only if $A + 1$ is stable.
h) True or false: A is stable if and only if A^2 is stable.
i) True or false: A is stable if $A^2 = 0$.
j) True or false: A is unstable if $A^2 = A$.
k) True or false: A is stable if A is diagonalizable.

Stability

A discrete dynamical system $x(t + 1) = Ax(t)$ is **asymptotically stable** if $x(t) \to 0$ as $t \to \infty$ for all initial conditions $x(0)$. (If we say "stable" we always mean asymptotically stable). The main result covered in this section is that a system is asymptotically stable if and only all eigenvalues of A have absolute value $|\lambda_j| < 1$. For example, a rotation dilation A with first column $Ae_1 = \begin{bmatrix} a \\ b \end{bmatrix}$ is stable if and only if $a^2 + b^2 < 1$. We often just say "A is stable" rather than "the origin is stable for the discrete dynamical system $x \mapsto Ax$".