Homework 29: Fourier I

This homework is due on Wednesday, April 18, respectively on Thursday, April 19, 2018.

1 a) Find the angle between the functions $f(x) = x^2$ and $g(x) = x^3$.
b) Project $f(x) = \sin^2(x)$ onto the plane spanned by $\sin(2x), \cos(2x)$.
c) Find the length of the function $f(x) = x^3$ in C_{per}^∞.

2 Verify that the functions $\cos(nx), \sin(nx), 1/\sqrt{2}$ form an orthonormal family.

3 Find the Fourier series of the function $f(x) = 11 + |6x|$.

4 Find the Fourier series of the function $4\cos(3x) + \sin^2(13x) + \sin(100x) + 10$.

5 Find the Fourier series of the function $f(x) = 10|\sin(x)|$.

Fourier Series I

We write 2π periodic functions always as functions on $[-\pi, \pi]$. In the space of piecewise smooth periodic functions C_{per}^∞ we have the inner product $\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^\pi f(x)g(x) \, dx$. This defines the **length** $|f| = \sqrt{\langle f, f \rangle}$ and angle $\cos(\alpha) = \langle f, g \rangle / (|f||g|)$. The functions $\sin(nx), \cos(nx), 1/\sqrt{2}$ form an orthonormal basis in C_{per}^∞ in the sense that they are linearly independent and span the space. Any function f can be written uniquely as a linear combination of these terms. With the Fourier coefficients $a_0 = \langle f, 1/\sqrt{2} \rangle, a_n = \langle f, \cos(nx) \rangle$ and $b_n = \langle f, \sin(nx) \rangle$, we can write

$$f = a_0/\sqrt{2} + \sum_n a_n \cos(nx) + \sum_n b_n \sin(nx)$$

This is called the Fourier series of f.

Example: Find the length of the function $f(x) = x^2$.
Solution: since

$$||f||^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^4 \; dx = 2\pi^4/5 \; ,$$

we have $||f|| = \sqrt{2/5}\pi^2$.

Example: find the Fourier series of the function $f(x) = x^2$ on $[-\pi, \pi]$.
Solution: the function is an even function. So, all the coefficients b_n are zero. We compute $a_0 = \frac{2}{\pi} \int_0^\pi x^2 1/\sqrt{2} \; dx = \sqrt{2}\pi^2/3$ and $a_n = \frac{2}{\pi} \int_0^\pi x^2 \cos(nx) \; dx = (4\cos(n\pi))/n^2 = 4(-1)^n/n^2$. The Fourier series is

$$a_0/\sqrt{2} + \sum_{n=1}^{\infty} a_n \cos(nx) = \pi^2/3 + \sum_{n=1}^{\infty} 4\cos(n\pi)/n^2 \cos(nx) \; .$$

Below you see the sum with 3 terms. It already gives quite a good fit of the parabola which is continued to a periodic function.