Homework 30: Fourier II

This homework is due on Friday, April 20, respectively on Tuesday, April 24, 2018.

1 Find the Fourier series of the function which is 8 on $[0, \pi/4]$ and zero everywhere else.

2 Use Parseval to find the length $|f| = \sqrt{(1/\pi) \int_{-\pi}^{\pi} f(x)^2 \, dx}$ for $f(x) = 2\cos(14x) + 4\cos(11x) + 2\sin(27x) - \cos(19x) + 5\cos(140x)$.

3 Compute both sides of the Parseval identity for $f(x) = x + |x|$.

4 Find $\Sigma_{n=1}^{\infty} \frac{1}{(2n)^2} = 1/4 + 1/16 + 1/36 + \dots$ from the known Basel problem formula of $\Sigma_n \frac{1}{n^2}$ and use this to compute the sum $\Sigma_{n=0}^{\infty} \frac{1}{(2n+1)^2}$ over the odd numbers.

5 This problem is a preparation for partial differential equations PDEs and consists of reminders. All statements are pretty straight forward. We work with functions on $[-\pi, \pi]$ described by Fourier series. You can refer to work done in the previous homework:
a) Verify that the Fourier basis $\mathcal{B} = \{1/\sqrt{2}, \cos(nx), \sin(nx)\}$ consists of eigenfunctions of D^2 on the space of piecewise smooth 2π-periodic functions.
b) What are the corresponding eigenvalues?
c) Show that every eigenfunction of D^2 is either constant or of the form $a\cos(nx) + b\sin(nx)$ for some n.
d) What are the eigenvalues of $D^2 + D^4 + 6$ on the subspace of C^{∞}_{per} consisting of odd functions?

Fourier Series II

Recall that the Fourier coefficients of a function $f \in C^\infty_{per}$ are defined as $a_0 = \langle f, 1/\sqrt{2} \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)/\sqrt{2}\ dx$, $a_n = \langle f, \cos(nt) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\ dx$, $b_n = \langle f, \sin(nt) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\ dx$. These are just inner products $a_n = \langle f, \cos(nx) \rangle$ and $b_n = \langle f, \sin(nx) \rangle$ and

$$f = a_0/\sqrt{2} + \sum_n a_n \cos(nx) + \sum_n b_n \sin(nx)$$

is the Fourier series of f. The Parseval identity is $||f||^2 = a_0^2 + \sum_{k=1}^{\infty} a_k^2 + b_k^2$. It is an extension of the Pythagoras theorem.