Name:

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MWF 10 Hunter Spink
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TTH 10 Ana Balibanu
TTH 10 Morgan Opie
TTH 10 Rosalie Belanger-Rioux
TTH 11:30 Philip Engel
TTH 11:30 Alison Miller

- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (20 points) No justifications are needed.

1)	T	F	The set of smooth functions in $C^\infty(R)$ with $f(2) = 3$ is a linear space.
2)	T	F	If matrix A is invertible, then rref(A) must be invertible too.
3)	T	F	If A is an invertible matrix, and $B =$ rref(A). Then $A^{-1} = B^{-1}$.
4)	T	F	There is a linear subspace of $\mathbf{R}^7$ that contains exactly seven vectors.
5)	T	F	There exists a 7×3 matrix that has rank 7.
6)	T	F	The circle $x^2 + y^2 = 1$ is the kernel of a linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$.
7)	T	F	If a matrix A is similar to a matrix B and A is invertible, then B is invertible.
8)	T	F	A reflection about the line $x + y = 1$ is a linear transformation.
9)	T	F	If $A^2BA^3 = I_3$ for 3×3 matrices A, B, then B is invertible.
10)	T	F	For any reflection A about the origin in R^2, there exists a 2×2 matrix B such that $A = B^2$.
11)	T	F	For any 2×2 matrix, we always have rank(A) = rank(A^2).
12)	T	F	It is possible that a system $Ax = b$ has a unique solution for some b if A is a 2×3 matrix.
13)	T	F	A reflection about the x-axis is similar to a rotation by 90 degrees in the plane.
14)	T	F	There is a 6×4 matrix for which the kernel has dimension 5.
15)	T	F	For any two 2×2 matrices A, B, the identity $(A - B)(A^2 + AB + B^2) = A^3 - B^3$ holds.
16)	T	F	The column vectors of a 2×2 rotation-dilation matrix form an orthogonal basis.
17)	T	F	If A is a non-invertible square matrix then rref(A) has at least one row of zeros.
18)	T	F	The plane $x + y + z = 1$ in space is the image of a linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^3$.
19)	T	F	For any $n \times n$ matrix A, the identity $\ker(A^3) = \ker(A^2)$ holds.
20)	T	F	If A is a 3×5 matrix of rank 3, then $A\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ has infinitely many solutions $\vec{x}$.

	Total

Problem 2) (10 points) No justifications are needed.

a) (5 points) Match the following transformations with their names. Choices can appear multiple times. A shear dilation is a shear composed with a scaling, a rotation dilation is a rotation composed with a scaling, a reflection dilation is a reflection composed with a scaling, a projection dilation is a projection composed with a scaling. The scaling factors can also be 1 of course.

Matrix	Enter A-D here.
a) $\begin{bmatrix} 4 & 3 \\ 3 & -4 \end{bmatrix}$	
b) $\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$	
c) $\begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}$	
d) $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$	
e) $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$	
f) $\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$	
g) $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$	
h) $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$	

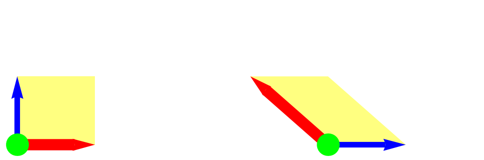
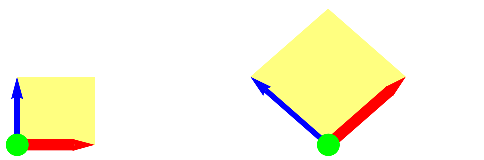
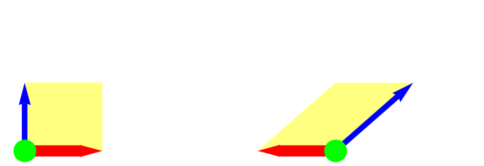
A) Reflection dilation

B) Shear dilation

C) Projection dilation

D) Rotation dilation

b) (5 points) Match the matrices with their actions:

A-J	domain / codomain	A-J	domain / codomain
			
			
			
			
			

A $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

B $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

C $\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$

D $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

E $\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$

F $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

G $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

H $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

I $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

J $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

Problem 3) (10 points) No justifications are needed.

a) (5 points) Check the boxes which apply.

matrix	similar to $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	invertible
$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$		
$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$		
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$		
$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$		
$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$		

b) (5 points) Which of the following sets are linear spaces, which are linear transformations, which of them are none?

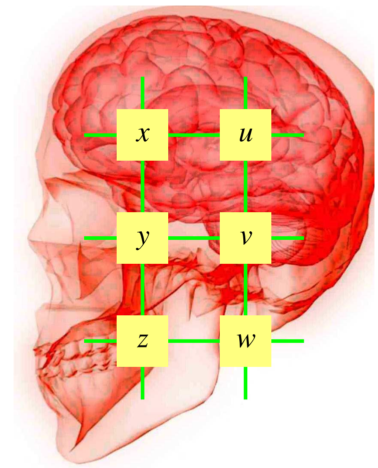
object	linear space	linear transformation
$T(x, y) = x + y$		
$\{(x, y) \mid x + y = 1\}$		
The identity matrix		
$B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$		
$\{(x, y) \mid x = y\}$		
The image the zero matrix		

Problem 4) (10 points)

Consider the system of linear equations

$$\left| \begin{array}{ccccccccccccc} x & & & & & + & u & & & & & = & 3 \\ & & y & & & & & + & v & & & = & 5 \\ & & & & z & & & & & + & w & = & 9 \\ x & + & y & + & z & & & & & & & = & 8 \\ & & & & & & u & + & v & + & w & = & 9 \end{array} \right|$$

Do we have infinitely many solutions, zero solutions or exactly one solution? If there are solutions, find all of them.



The problem appears in **tomography** like magnetic resonance imaging. A scanner can measure averages of tissue densities along lines. The task is to compute the actual densities.

Problem 5) (10 points)

Let A be the matrix of a reflection dilation $T(x) = T_2(T_1(x))$, where the reflection T_1 is done at the line $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and where the scaling is $T_2(x) = 2x$.

a) (5 points) Find a suitable basis for this problem in which the transformation is given by a diagonal matrix B.

b) (5 points) Find the matrix A of the transformation T in the standard basis.

Problem 6) (10 points)

Find a basis of the image and kernel of the following matrix and state what the rank-nullety theorem tells in this situation.

$$A = \begin{bmatrix} 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ 2 & 1 & 0 & 0 & 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 0 & 0 & 1 & 2 \\ 2 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{bmatrix} .$$

Problem 7) (10 points)

Find a matrix A such that the image of the matrix $B = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}$ coincides with the kernel of A.

Problem 8) (10 points)

Describe the transformation $T(x) = Ax$ with

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

in the basis $\mathcal{B}$ given by the column vectors of the matrix

$$S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} .$$

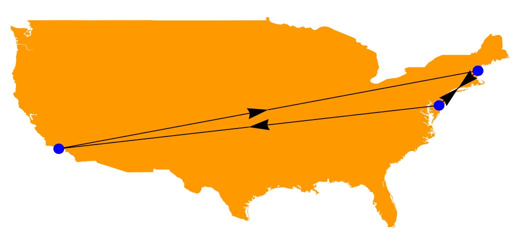
In other words, find the 3×3 matrix B which describes T in the $\mathcal{B}$ coordinates.

Problem 9) (10 points)

An airline services Boston, New York, Los Angeles. It flies from New York to Los Angeles, from Los Angeles to Boston and from Boston to New York as well as from New York to Boston.

The connection matrix is

$$\begin{bmatrix} & BO & NY & LA \\ BO & 0 & 1 & 0 \\ NY & 1 & 0 & 1 \\ LA & 1 & 0 & 0 \end{bmatrix} .$$



a) (7 points) To find the number of different round trips of length 8 starting from Boston, one can compute $A^8 = A \cdot A \cdot A \cdot A \cdot A \cdot A \cdot A \cdot A$ with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

and to look up the first entry in the first row of A^8. Compute the matrix A^8 and find so the number of round trips of length 8.

b) (3 points) Find the 2×2 matrix B such that

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} B \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} .$$

Hint to a). Compute first $U = A \cdot A$, then $V = U \cdot U$ and finally $A^8 = V \cdot V$.

Problem 10) (10 points)

A **two person zero-sum game** for two players Ana and Bob (Mathematicians like palindromes) is described by the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 5 \end{bmatrix}$$

The interpretation is that player "Ana" has two possible moves and player "Bob" has 3 possible moves. If Ana for example makes the first move and Bob makes the 2nd move, then the payoff for Bob to Ana is $A_{12} = 2$. If the row vector $a = [a_1 a_2]$ encodes the probabilities that Ana makes the moves and the column matrix $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ encodes the probability of the moves of Bob, then the **expected payoff** is

$$E(a, b) = aAb .$$

a) (4 points) Assume Ana choses her moves with equal probability $1/2, 1/2$ and Bob choses his decisions with equal probability $1/3, 1/3, 1/3$. What is the expected payoff?

b) (3 points) Assume Bob fixes his strategy and keeps his equal probability move strategy from a). What strategy a maximizes the expected payoff for Ana?

c) (3 points) Assume now Ana fixes her fifty/fifty strategy. What strategy b minimizes the expected payoff Bob has to pay to Ana?