Name:

MWF 9 Oliver Knill
MWF 10 Jeremy Hahn
MWF 10 Hunter Spink
MWF 11 Matt Demers
MWF 11 Yu-Wen Hsu
MWF 11 Ben Knudsen
MWF 11 Sander Kupers
MWF 12 Hakim Walker
TTH 10 Ana Balibanu
TTH 10 Morgan Opie
TTH 10 Rosalie Belanger-Rioux
TTH 11:30 Philip Engel
TTH 11:30 Alison Miller

- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (20 points) No justifications needed

1)	T	F	The plane $x + y + z - 1 = 0$ is the kernel of a linear transformation T.
2)	T	F	For any matrix A the identity $\mathrm{im}\,(A^2) = \mathrm{im}\,(\mathrm{rref}\,(A^2))$ holds.
3)	T	F	For any matrix A, one has $\ker(A) = \ker(\mathrm{rref}\,(A))$.
4)	T	F	There is a 4×8 matrix whose kernel is 3-dimensional.
5)	T	F	There is a 2×2 matrix for which $A^2 = -I_2$.
6)	T	F	If three vectors v_1, v_2, and v_3 are in a plane, they are independent.
7)	T	F	If S and A are invertible $n \times n$ matrices, then $(SAS^{-1})^{-1} = S^{-1}A^{-1}S$.
8)	T	F	If all entries of a 2×2 matrix A are nonzero, then the inverse of A exists.
9)	T	F	For any square matrix A, the image of A^7 is contained in the image of A.
10)	T	F	If A is a matrix, let B be the matrix for which the order of the columns are reversed. Then $\mathrm{rref}\,(A) = \mathrm{rref}\,(B)$.
11)	T	F	If the columns of a $n \times n$ matrix form a basis in $\mathbb{R}^n$, then the rows also form a basis in $\mathbb{R}^n$.
12)	T	F	If $B^2 = A$, then B is called the square root of A. Every 2×2 matrix A has either 0 or 1 or 2 square roots.
13)	T	F	If A is an invertible 2×2 matrix and $\mathcal{B}$ is the basis of the column vectors, then $[A]_{\mathcal{B}}$ is diagonal.
14)	T	F	There exists a linear transformation whose image consists of exactly 6 distinct points.
15)	T	F	$\mathcal{B} = \{2e_1, 2e_2\}$ is a basis of $\mathbb{R}^2$ for which $[v]_{\mathcal{B}} = v$ for any v.
16)	T	F	The dimension of the image of a matrix A is equal to the dimension of the image of the matrix $\mathrm{rref}(A)$.
17)	T	F	There exists an invertible $n \times n$ matrix whose inverse has rank $n - 1$.
18)	T	F	If A is the reflection about a line L and B is the reflection about the plane $V = L^\perp$ perpendicular to L, then $A = -B$.
19)	T	F	The set of vectors in space satisfying $x + y + z = 1$ form a linear space of dimension 2.
20)	T	F	If A, B are given $n \times n$ matrices, then there is a unique $n \times n$ matrix X satisfying $(A + X)B = A$ if B is invertible.

	Total

Problem 2) (10 points)

Match the matrices to their rref, enter R, U, V, W in the right order below:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad V = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad W = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

rref $(A)=$	rref $(B)=$	rref $(C)=$	rref $(D)=$

b) Find the projection matrix onto the space spanned by the vectors $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}/2$, $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}/2$.

Problem 3) (10 points)

Each of the following matrices matches with a transformation below:

$$A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ -\sqrt{2} & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$D = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \quad E = \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix} \quad F = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$$

rotation	dilation	rotation dilation	shear	projection	reflection

Each of the following spaces R, U, V, W below is equal to one of the spaces K, L, M, N. No justification is necessary.

$$R = \text{im}\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \quad U = \text{im}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \quad V = \text{im}\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad W = \text{im}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$K = \ker\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \quad L = \ker\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$M = \ker\begin{bmatrix} 0 & 1 & -1 & 0 \\ 2 & -1 & -1 & 2 \end{bmatrix} \quad N = \ker\begin{bmatrix} 0 & 2 & -2 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix}$$

R =	U =	V =	W =

Problem 4) (10 points)

Solve the following system of linear equation by doing row reduction of the augmented matrix. In each step you have to use one of three basic row reduction steps. Write in each step, what you do:

$$\begin{array}{ccccccc} x & & +z & & = & 1 \\ & y & +z & +q & = & 2 \\ x & & +z & +q & = & 2 \\ x & +y & & +q & = & 1 \end{array}$$

Problem 5) (10 points)

Two matrices A, B are called **similar** if there exists an invertible matrix S such that $B = S^{-1}AS$.

a) (2 points) Is it possible that the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ be similar to a reflection about a line? Show your reasoning.

b) (3 points) What is the matrix A in the basis $\mathcal{B} = \{v_1, v_2, v_3\} = \{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \}$?

c) (3 points) Find the inverse of the matrix S which contains the above basis as column vectors v_1, v_2, v_3. Use row reduction to find the inverse.

d) (2 points) Express $\vec{v} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$ as a linear combination of v_1, v_2 and v_3.

Problem 6) (10 points)

Here e_1, e_2, e_3 denote the standard basis vectors in $\mathbb{R}^n$.

a) (3 points) Find the 3×3 matrix A which is the reflection about the xz-plane.

b) (2 points) Find the 3×3 matrix B which maps e_1 to e_2, e_2 to e_3 and e_3 to e_1.

c) (2 points) Find the 3×3 matrix C which scales every vector by a factor 2.

d) (3 points) What is the transformation CBA which first reflects, then rotates and finally scales.

Problem 7) (10 points)

Find a basis for the image and the kernel of the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 2 & 3 & 4 & 3 & 2 \\ 3 & 4 & 5 & 4 & 3 \\ 4 & 5 & 6 & 5 & 4 \\ 5 & 6 & 7 & 6 & 5 \end{bmatrix} .$$

Problem 8) (10 points)

We are given the rotation dilation matrix $A = \begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$.

a) (3 points) Find a matrix B such that $B^2 = A$.

b) (4 points) Write down the matrix B^{17}. We need a numerical result which can involve powers of numbers.

c) (3 points) Can A be the product of a projection at a line L and a reflection Q at a second line K?

Problem 9) (10 points)

Let A be a 3×3 matrix such that $A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

a) (3 points) Why is $\text{im}\,(A^2)$ a subspace of $\ker\,(A)$?

b) (4 points) Find all possible values for $\text{rank}(A)$.

c) (3 points) Give an example of such a matrix A for each possible rank.

Problem 10) (10 points)

Let A be a 3×3 matrix with $A^3 = 0$, the zero matrix.

a) (3 points) Compare $\ker\,(A)$ and $\text{im}\,(A)$ and $\text{im}\,(A^2)$. How are they related?
b) (2 points) Is A invertible?
c) (2 points) Is $A + I_3$ invertible where I_3 is the identity matrix? Hint. Look at $B = I_3 - A + A^2$.
d) (3 points) Give examples of matrices A exhibiting all the relations you found in part a) and c).