Name:

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MWF 10 Jeremy Hahn
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MWF 11 Yu-Wen Hsu
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TTH 10 Ana Balibanu
TTH 10 Morgan Opie
TTH 10 Rosalie Belanger-Rioux
TTH 11:30 Philip Engel
TTH 11:30 Alison Miller

- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (20 points) No justifications are needed.

1) T F The rank of A^{-1} is always equal to the rank of A if A is an invertible matrix.

Solution:
It has to have full rank in order that it is invertible.

2) T F $\text{rank}(A - B) = \text{rank}(A) - \text{rank}(B)$ for all 2×2 matrices.

Solution:
Take $A = B = 1_n$.

3) T F The row reduced echelon form of an invertible 3×3 matrices is invertible.

Solution:
It is the identity

4) T F Given 3 vectors in R^5, then their span forms a linear space.

Solution:
It is the image of a matrix.

5) T F A system of linear equations has either 0, 1 or ∞ many solutions.

Solution:
This is an important property for systems of linear equations.

6) T F A reflection in the plane at the x axes is similar to the reflection at the y axes.

Solution:
Just take $S(e_1) = e_2, S(e_2) = e_1$. This conjugates the two reflections.

7) T F Every basis of R^3 contains exactly 3 vectors in it.

Solution:
The number of basis vectors is called the dimension. We have seen that this number does not depend on the choice of the basis.

8) T F A 4×4 matrix can have $\dim(\mathrm{im}(A)) = \dim(\ker(A))$.

Solution:

An example is $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

9) T F The rank of a 7×3 matrix can be 4.

Solution:
We can not have more than 3 leading 1's because each leading one is in its own column.

10) T F If $\{v_1, v_2, v_3, v_4\}$ is a set of vectors spanning a linear subspace V of $\mathbf{R}^9$, then $\dim(V) \geq 4$.

Solution:
It can also be smaller. We can for example take $v_1 = v_2 = v_3 = v_4$ in which case the linear subspace is 1 dimensional.

11) T F If A is a 7×5 matrix, then the dimension of $\ker(A)$ is at least 2.

Solution:
The nulletly can be 0 since we can have a leading 1 in $\mathrm{rref}(A)$ in every column.

12) T F The difference $A - B$ of 2 invertible 5×5 matrices A, B is invertible.

Solution:
We can take $A = I_n$ and $B = I_n$. Then the difference is not invertible.

13) | T | F | If $A\vec{x} = \vec{0}$ has a nonzero solution, where A is a 4×4 matrix, then $\text{rank}(A) \leq 3$.

Solution:
The kernel is at least 1 dimensional. The rank-nullity theorem implies that the image is maximally 3 dimensional.

14) | T | F | If $\vec{b}$ is in $\text{im}(A)$, then $A\vec{x} = \vec{b}$ has exactly one solution.

Solution:
There can be a kernel.

15) | T | F | If A and B are 2×2 matrices and $A \cdot B$ is the identity matrix 1_2, then A and B are both invertible.

Solution:
Yes.

16) | T | F | If $\vec{v}$ is a nonzero vector in the kernel of A, then $\vec{v}$ is perpendicular to every row vector of A.

Solution:
This is what $A\vec{v} = \vec{0}$ means.

17) | T | F | If $AB = I_2$ for an 2×3 matrix A and B is a 3×2 matrix, then $BA = I_3$.

Solution:
This is already not true for $A = [1, 0], B = [1, 0]^T$ where $AB = I_1$ and BA is a projection.

18) | T | F | If a 2×2 matrix different from the identity is its own inverse then it is a reflection at a line.

Solution:
It can be a reflection at a point.

19) | T | F | The set of vectors (x, y) in R^2 such that $|x| + y = 0$ is a linear subspace of R^2.

Solution:
It is not closed under scalar multiplication.

20) | T | F | Given a rotation dilation matrix A and a projection matrix B. Then the intersection of the image of A and the kernel of B is a linear space.

Solution:
The intersection of two linear spaces is a linear space.

[ ] Total

Problem 2) (10 points) No justifications are needed.

a) (5 points) Which of the following matrices are in row reduced echelon form?

Matrix	is in row reduced echelon form
$\begin{bmatrix} 1 & 2 & 3 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	
$\begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$	
$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$	
$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$	
$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$	

b) (5 points) Check the matrices which are invertible:

Matrix	invertible
$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	
$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$	
$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}$	
$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	
$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$	

Solution:
a) The first 2 are in row reduced echelon form.
b) All except the third one are invertible.

Problem 3) (10 points) No justifications are necessary.

a) (3 points) Which of the following matrices either perform a rotation dilation or a reflection dilation? Check the corresponding boxes (it is also possible that both cases are unchecked):

Matrix	rotation dilat.	reflection dilat.	Matrix	rotation dilat.	reflection dilat.
$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$			$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$		
$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$			$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$		

Solution:

Matrix	rotation dilat.	reflection dilat.	Matrix	rotation dilat.	reflection dilat.
$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$	X		$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$		X
$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$			$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$		

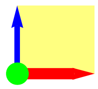
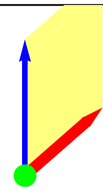
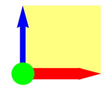
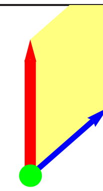
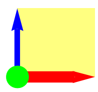
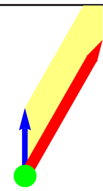
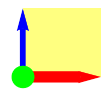
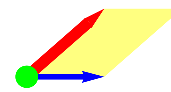
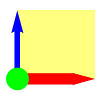
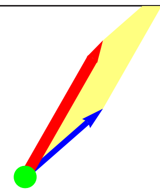
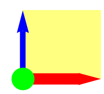
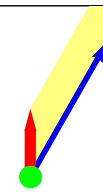
b) (2 points) Which of the following sets are linear spaces?

The space of all ...	Check
all (x, y) satisfying $x^2 + y^2 = 1$	
all (x, y) satisfying $x^2 = y$	

The space of all ...	Check
$(x, y, z) \in \mathbf{R}^3$ satisfying $2x + y - 4z = 1$	
the set of rational numbers in R	

Solution:
Solution to b) No, No, No, NoooooO! To see why, always follow protocol! Check first whether the zero vector is there. This fails for the circle and the plane not going through the origin. For the parabola, the scaling condition fails $(1, 1)$ is there but $(3, 3)$ not. Also for the rational numbers, the scaling condition fails. You can not multiply a given vector like 1 which is in the space with the real number π for example.

c) (5 points) Match the matrices with the action of the transformation which maps a shape in the domain $\mathbb{R}^2$ into a shape of the codomain $\mathbb{R}^2$.

A-F	domain	codomain	A-F	domain	codomain
					
					
					

A= $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

B= $\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$

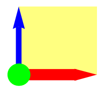
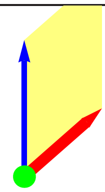
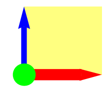
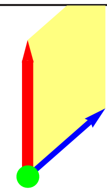
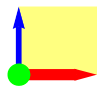
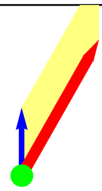
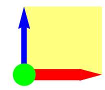
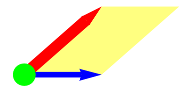
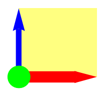
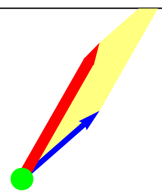
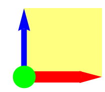
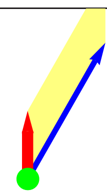
C= $\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$

D= $\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$

E= $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

F= $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$

Solution:

A-F	domain	codomain	A-F	domain	codomain
B			D		
A			E		
F			C		

Problem 4) (10 points)

a) (5 points) Find a basis of the image of the following **chess matrix**:

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} .$$

b) (5 points) Find a basis for the linear subspace of all vectors in $\mathbf{R}^4$ which are perpendicular to the columns of the matrix

$$A = \begin{bmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{bmatrix} .$$

Solution:
a) We have to find the kernel of A^T! By accident this is A. Row reduction gives

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

There are two leading 1, the first two columns of the original matrix therefore form the basis for the image. The answer is

$$\mathcal{B} = \{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} , \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \} .$$

b) Row reduction of the transpose A gives

$$A = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} .$$

By accident, this is also the row reduction of A. But it was important to row reduce the transpose. We see two free variables in the last two rows and get $x-s-2t = 0, y+2s+3t = 0, z = s, w = t$ leading to the kernel

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = s \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

A basis for the kernel is

$$\mathcal{B} = \{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} , \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \} .$$

Problem 5) (10 points)

a) (5 points) Invert the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

by using row reduction on an augmented 4×8 matrix

b) (5 points) Find a basis for the linear space of vectors perpendicular to the kernel of

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} .$$

Solution:
a) To get the inverse, we row reduce the augmented matrix

$$A = \left[\begin{array}{cccc|cccc} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

A good start is to place the last row on the top:

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right] .$$

Then subtract the third row from the second row

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right] .$$

Finally subtract the last row from the second last

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right] .$$

We can now see the inverse:

$$A^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} .$$

b) To get the kernel of A, we row reduce A and get

$$\text{rref}(A) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} .$$

We see one leading 1 and have therefore, one free variable t. The equations $x+t = 0, y = t$ show that the kernel is the linear space spanned by $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. A basis for the space of vectors perpendicular to the kernel is the kernel of

$$\begin{bmatrix} 1 & -1 \end{bmatrix} .$$

which is spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Problem 6) (10 points)

a) (2 points) Find the angle between the vectors $x = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}$ and $y = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix}$.

b) (2 points) Find a matrix A whose image is spanned by the two vectors x, y.

c) (4 points) Find a matrix B whose image is the space of all vectors perpendicular to the two vectors.

d) (2 points) What is the relation between the image of B and the kernel of the transpose of A?

Solution:
a) $\cos(\alpha) = 1/2$. The angle is $\pi/3$.

b) $A = \begin{bmatrix} 2 & 2 \\ 0 & 0 \\ 2 & 0 \\ 0 & 2 \end{bmatrix}$.

c) Put the columns of A into the rows of a matrix C and and row reduce to find the kernel. This gives $\text{rref}(C) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$. We have $x_1 = -t, x_2 = s, x_3 = t, x_4 = t$, so that $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ span the kernel of B. Stick them into the columns of a matrix to get $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$ works. d) The image of B is equal to the kernel of the transpose of A.

Problem 7) (10 points)

a) (6 points) Find a basis of the space V of all vectors perpendicular to the three vectors

$$\{v_1, v_2, v_3\} = \{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \} .$$

b) (4 points) Use a) to find a basis for R^4 which contains v_1, v_2, v_3.

Solution:
a) Define

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \} .$$

The row reduction of A is

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \} .$$

We see that there is one free variable. Writing this down gives $x = 0, y = 0, z = -s, w = s$ so that $v = [0, 0, -1, 1]$ is a basis for the kernel.
b) We can take the vectors $\{v_1, v_2, v_3, v\}$, where v was obtained in a).

Problem 8) (10 points)

a) (5 points) The projection-dilation matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ in the basis

$$\mathcal{B} = \{v_1, v_2, v_3\} = \{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \}$$

is given by a matrix B. Find this 3×3 matrix B.

b) (5 points) A linear transformation T satisfies

$$T(v_1) = v_2, T(v_2) = v_3, T(v_3) = v_1$$

where v_1, v_2, v_3 are given in a). Find the matrix R implementing this transformation in the standard basis.

Solution:
a)

$$S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$B = S^{-1}AS = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} .$$

b) $B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $R = SBS^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 2 & -1 \end{bmatrix}$.

Problem 9) (10 points)

a) (5 points) Find A^{10} where $A = \begin{bmatrix} 4 & 3 & 0 & 0 \\ 3 & -4 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

b) (5 points) Find a 2×2 matrix X satisfying $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} X \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}$

Solution:

a) You might spot the reflection dilation $B = \begin{bmatrix} 4 & 3 \\ 3 & -4 \end{bmatrix}$ and a projection dilation $C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. We have $B^2 = 25I_2$ and $C^2 = 2C$. This gives $B^{10} = 25^5 \cdot I_2$ and $C^{10} = 2^9 \cdot C$. Therefore,

$$A^{10} = \begin{bmatrix} 25^5 & 0 & 0 & 0 \\ 0 & 25^5 & 0 & 0 \\ 0 & 0 & 2^9 & 2^9 \\ 0 & 0 & 2^9 & 2^9 \end{bmatrix}$$

b) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} X \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ -2 & 3 \end{bmatrix}$.

Now multiply from the right with $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1}$ and from the left with $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^{-1}$ to isolate $X = \begin{bmatrix} 1/2 & 2 \\ -5/3 & 1 \end{bmatrix}$.

Problem 10) (10 points)

The data points $(1,2), (2,2), (-3,-4)$ in the plane relate two vectors $X = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ and $Y = \begin{bmatrix} 2 \\ 2 \\ -4 \end{bmatrix}$.

a) (2 points) Find the length of X and Y. (Since the vectors are already centered these are up to a constant the standard deviation)

b) (2 points) Find the correlation coefficient, the cos of the angle between X and Y.

c) (4 points) The regression line $y = ax$ is given by the formula $a = X \cdot Y/|X|^2$. Find a.

d) (2 points) What can you deduce from c): is the angle between X and Y acute or obtuse?

Solution:

a) $\sqrt{14}$ and $\sqrt{24}$.

P.S. About the statistics connection: as we did not want to get into normalization things or get into discussions why in statistics, one divides by $\sqrt{n-1}$ rather than $\sqrt{n}$, we have kept the normalizations away also when referencing to statistics. In the definition of the correlation coefficient, statisticicians use the normalization $Cov[X,Y]$ with n rather than $n-1$. There are reasons from estimation theory why dividing by $n-1$ is better than n in the standard deviation. **if one does not know the mean** but this solution of a variational problem is rarely significant in real life as statistics with data for which this is significant is otherwise questionable. By the way, it is better to divide by n if one knows the mean. And also this is part of estimation theory and a mathematical theorem that it is best. In any way, the picture of the correlation coefficient as the cos of an angle between two vectors is exactly what is done in statistics. The normalization does not matter. b) $\cos(\alpha) = X \cdot Y/14 = 18/(\sqrt{14}\sqrt{24})$. This number does not depend on the normalization.

c) $a = 9/7$. Also this number does not depend on the normalization. Keeping things geometric has lots of advantages as you can see when you look up formulas for the correlation coefficient or slope of the correlation line in statistics cookbooks.

d) Acute angle (means positive correlation).