STOKES THEOREM II | **Maths21a, O. Knill**

curl: $\text{curl}(M,N,P) = (P_y - N_z, M_z - P_x, N_x - M_y)$.
gradient: $\text{grad}(f) = (f_x, f_y, f_z)$.
Fundamental theorem of line integrals
curve $r(t) : [a,b] \to C$ with boundary $\{r(a), r(b)\}$.

$$\int_a^b \text{grad}(f)(r(t)) \cdot r'(t)\ dt = f(r(b)) - f(r(a))$$

Stokes theorem:
surface $r(u,v) : R \to S$ with boundary $r(t) : [a,b] \to C$

$$\int\int_R \text{curl}(F)(r(u,v)) \cdot (r_u \times r_v) dudv = \int_a^b F(r(t)) \cdot r'(t)\ dt$$

REMINDERS:

CURL: $\boxed{\text{curl}(F) = \nabla \times F}$

GRAD: $\boxed{\text{grad}(f) = \nabla f}$

FTL: $\boxed{\int_C \text{grad}(f)\ \cdot dr = f(B) - f(A)}$

STOKES: $\boxed{\int\int_S \text{curl}(F)\ \cdot dS = \int_C F \cdot dr}$

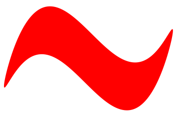
CURL(GRAD)=0: $\boxed{\nabla \times \nabla f = \vec{0}}$

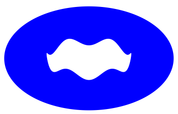
CONSERVATIVE FIELDS. F is **conservative** if is a gradient field $F = \nabla f$. The fundamental theorem of line integral implies that the line integral along closed curves is zero and that line integrals are path independent. We have also seen that $F = \text{grad}(f)$ implies that F has zero curl. If we know that F is conservative, how do we compute f? If $F = (M,N,P) = (f_x, f_y, f_z)$, we got f by integration. There is an other method which you do in the homework in two dimensions: to get the potential value f, find a path C_P from the origin to the point $P = (x,y,z)$ and compute $\int_C F \cdot dr$. Because line integrals are path independent, the fundamental theorem of line integrals gives $\int_C F \cdot dr = f(P)$.

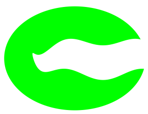
CONNECTED. A region is called **connected** if one can connect any two points in the region with a path.

SIMPLY CONNECTED. A region is called **simply connected** if it is connected and every path in the region can be deformed to a point within the region.

EXAMPLES 2D:

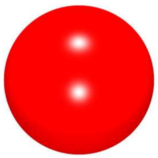

Simply connected.


Not simply connected.

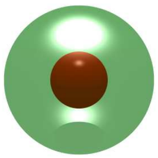

Simply connected.

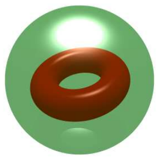

Not simply connected.

EXAMPLES 3D:


A solid ball is simply connected.


A solid torus is not simply connected.

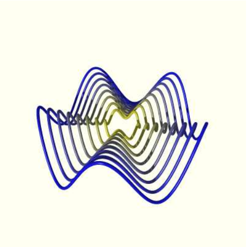

The complement of a solid ball is simply connected.


The complement of a solid torus is not simply connected.

THEOREM.

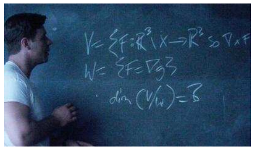
> In a simply connected region D, a vector field F is conservative if and only if $\text{curl}(F) = \vec{0}$ everywhere inside D.

Proof. We already know that $F = \nabla f$ implies $\text{curl}(F) = \vec{0}$. To show the converse, we verify that the line integral along any closed curve C in D is zero. This is equivalent to the path independence and allows the construction of the potential f with $F = \nabla f$.
By assumption, we can deform the curve to a point: if $r_0(t)$ is the original curve and $r_1(t)$ is the curve $r_1(t) = P$ which stays at one point, define a parametrized surface S by $r(t,s) = r_s(t)$. By assumption, $\text{curl}(F) = \vec{0}$ and therefore the flux of $\text{curl}(F)$ through S is zero. By Stokes theorem, the line integral along the boundary C of the surface S is zero too.



THE NASH PROBLEM. Nash challenged his multivariable class in the movie "A beautiful mind" with a problem, where the region is not simply connected.

> Find a region X of $\mathbf{R}^3$ with the property that if V is the set of vector fields F on $\mathbf{R}^3 \setminus X$ which satisfy $\text{curl}(F) = 0$ and W is the set of vector fields F which are conservative: $F = \nabla f$. Then, the space V/W should be 8 dimensional.



You solve this problem as an inclass exercise (ICE). The problem is to find a region D in space, in which one can find 8 different closed paths C_i so that for every choice of constants $(c_1, ..., c_8)$, one can find a vector field F which has zero curl in D and for which one has $\int_{C_1} F\ dr = c_1, ..., \int_{C_8} F\ dr = c_8$.

One of the many solutions is cut out 8 tori from space. For each torus, there is a vector field F_i (a vortex ring), which has its vorticity located inside the ring and such that the line integral of a path which winds once around the ring is 1. The vector field $F = c_1 F_1 + ... + c_8 F_8$ has the required properties.

CLOSED SURFACES. Surfaces with no boundaries are called **closed surfaces**. For example, the surface of a donought, or the surface of a sphere are closed surfaces. A half sphere is not closed, its boundary is a circle. Half a doughnut is not closed. Its boundary consists of two circles.

THE ONE MILLION DOLLAR QUESTION. One of the Millenium problems is to determine whether any three dimensional space which is simply connected is deformable to a sphere. This is called the **Poincare conjecture**.

LINEINTEGRAL IN HIGHER DIMENSIONS. Line integrals are defined in the same way in higher dimensions. $\int_C F \cdot dr$, where $\cdot$ is the dot product in d dimensions and $dr = r'(t)dt$.

CURL IN HIGHER DIMENSIONS. In d-dimensions, the curl is the field $\text{curl}(F)_{ij} = \partial_{x_j} F_i - \partial_{x_i} F_j$ with $\binom{d}{2}$ components. In 4 dimensions, it has 6 components. In 2 dimensions it has 1 component, in 3 dimensions, it has 3 components.

SURFACE INTEGRAL IN HIGHER DIMENSIONS. In d dimensions, a surface element in the ij-plane is written as dS_{ij}. The flux integral of the curl of F through S is defined as $\int\int \text{curl}(F) \cdot dS$, where the dot product is $\sum_{i<j} \text{curl}(F)_{ij} dS_{ij}$. If S is given by a map X from a domain R in the 2-plane to R^d, $U = \partial_u X$ and $V = \partial_v X$ are tangent vectors to that plane and $dS_{ij}(u,v) = (U_i V_j - U_j V_i)\ dudv$.

STOKES THEOREM IN HIGHER DIMENSIONS. If S is a two dimensional surface in d-dimensional space and C is its boundary, then $\int\int_S \text{curl}(F) \cdot dS = \int_C F \cdot ds$.