

1) TRUE/FALSE (circle one)

a) Orthogonal projection onto a subspace is an orthogonal transformation.	True or <input checked="" type="radio"/> False
b) If A is any square matrix, then $\det(-A) = -\det(A)$.	True or <input checked="" type="radio"/> False
c) If A is an invertible matrix that is <u>similar</u> to its own inverse A^{-1} , then $\det(A)$ must be either $+1$ or -1 .	<input checked="" type="radio"/> True or False
d) Let A be an orthogonal matrix. Then $\det(A) = 1$.	True or <input checked="" type="radio"/> False
e) If S and A are orthogonal $n \times n$ matrices, then the matrix $S^{-1}AS$ is orthogonal as well.	<input checked="" type="radio"/> True or False
f) It is always the case that $\det(A^T A) \geq 0$.	<input checked="" type="radio"/> True or False

- Ⓐ Orthogonal transformations must preserve length. Projections don't.
- Ⓑ Determinant is linear in any given row or column, but it's not linear.
If A is an $n \times n$ matrix, then $\det(-A) = (-1)^n \det A$.
- Ⓒ If A is similar to A^{-1} , then $\det(A) = \det(A^{-1})$. However, $\det(A^{-1}) = \frac{1}{\det A}$.
So $\det A = \frac{1}{\det A} \Rightarrow (\det A)^2 = 1 \Rightarrow \det A = \pm 1$.
- Ⓓ If A is orthogonal, then either $\det(A) = 1$ or $\det(A) = -1$.
- Ⓔ If S preserves norm, then so does S^{-1} , and likewise for the composition $S^{-1}AS$.
- Ⓕ In fact, if $A = \begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_k \end{bmatrix}$, k -volume of $\{\vec{v}_1, \dots, \vec{v}_k\}$ will be $\sqrt{\det(A^T A)}$.

2) Let P_2 is the linear space consisting of all polynomials of degree ≤ 2 , and let $T: P_2 \rightarrow P_2$ be defined by $T(f) = f'' - 2f' + 3f$.

a) Find the matrix of this linear transformation relative to the basis $\{1, t, t^2\}$.

$$f(t) = a + bt + ct^2$$

$$f'(t) = b + 2ct$$

$$f''(t) = 2c$$

$$\begin{aligned} [T(f)](t) &= [2c] - 2[b + 2ct] + 3[a + bt + ct^2] \\ &= (3a - 2b + 2c) + (3b - 4c)t + 3ct^2 \end{aligned}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \xrightarrow{T} \begin{bmatrix} 3a - 2b + 2c \\ 3b - 4c \\ 3c \end{bmatrix} = \begin{bmatrix} 3 & -2 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

MATRIX IS

$$\begin{bmatrix} 3 & -2 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & 3 \end{bmatrix}$$

b) Find the determinant of this linear transformation.

Determinant is same rel. to any basis, so use above matrix:

$$\det(T) = 3 \cdot 3 \cdot 3 = \underline{\underline{27}}$$

c) Is this linear transformation invertible, i.e. an isomorphism? Briefly explain.

$\det(T) \neq 0$, so T is invertible.

d) Using the above information, find a quadratic polynomial $f(t)$ such that $f'' - 2f' + 3f = 3t^2$.

want f such that $T(f) = 3t^2$. If $f(t) = a + bt + ct^2$,

then, relative to $\{1, t, t^2\}$ basis, we want

$$\begin{bmatrix} 3 & -2 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}. \text{ This can be solved by row reduction:}$$

$$\left[\begin{array}{ccc|c} 3 & -2 & 2 & 0 \\ 0 & 3 & -4 & 0 \\ 0 & 0 & 3 & 3 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 3 & -2 & 2 & 0 \\ 0 & 3 & -4 & 0 \\ 0 & 0 & 1 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 3 & -2 & 0 & -2 \\ 0 & 3 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 3 & -2 & 0 & -2 \\ 0 & 1 & 0 & 4/3 \\ 0 & 0 & 1 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 3 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & 4/3 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2/9 \\ 0 & 1 & 0 & 4/3 \\ 0 & 0 & 1 & 1 \end{array} \right] \Rightarrow \begin{aligned} a &= 2/9 \\ b &= 4/3 \\ c &= 1 \end{aligned} \Rightarrow \boxed{f(t) = \frac{2}{9} + \frac{4}{3}t + t^2}$$

3) Consider the following inconsistent system of linear equations: $\begin{cases} 2x + y = 2 \\ x - y = 1 \\ x + 2y = -2 \end{cases}$.

a) Find the least-squares solution for this linear system. [Note: This is not a data-fitting problem.]

$$\begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \quad A\vec{x} = \vec{b} \Rightarrow A^T A \vec{x} = A^T \vec{b}$$

$$\begin{bmatrix} 6 & 3 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

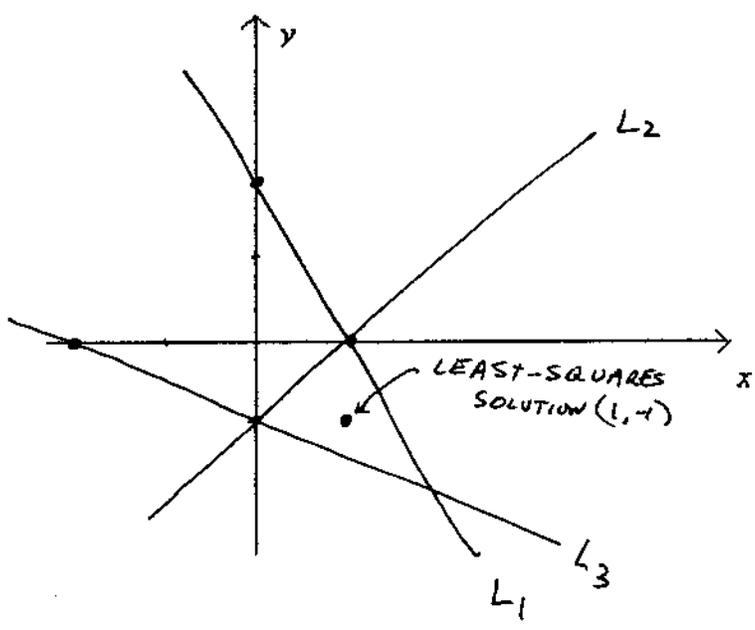
$$A^T A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 3 & 6 \end{bmatrix}$$

$$A^T \vec{b} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 3 & | & 3 \\ 3 & 6 & | & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & | & -1 \\ 6 & 3 & | & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & | & -1 \\ 0 & 9 & | & -9 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & | & -1 \\ 0 & 1 & | & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & | & 1 \\ 0 & 1 & | & -1 \end{bmatrix} \Rightarrow \boxed{(x, y) = (1, -1)}$$

b) Each equation in the given system represents a line in \mathbb{R}^2 . Describe in words and/or pictures the relationship between these three lines and the point you found in part a).



The Least-Squares Solution is at the center of the triangle formed by the three lines.

4) We are given three vectors in \mathbf{R}^4 : $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$.

a) Find the area of the parallelogram determined by the vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \\ -1 & 0 \end{bmatrix} \quad A^T A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$$

$$\text{Area} = \sqrt{\det(A^T A)} = \sqrt{4} = 2$$

b) Construct an orthonormal basis for the two-dimensional subspace of \mathbf{R}^4 spanned by $\{\mathbf{v}_1, \mathbf{v}_2\}$.

Call the vectors of this orthonormal basis \mathbf{w}_1 and \mathbf{w}_2 .

$$\|\vec{v}_1\| = \sqrt{1+1+1+1} = 2 \quad \vec{v}_2 - (\vec{v}_2 \cdot \vec{w}_1) \vec{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2}(2) \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$\vec{w}_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$

NORMALIZE \Rightarrow $\frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \vec{w}_2$
 (ALREADY A UNIT VECTOR)

c) Find the orthogonal projection of \mathbf{v}_3 in the subspace spanned by the vectors \mathbf{v}_1 and \mathbf{v}_2 .

$$\text{Proj}_V(\vec{v}_3) = (\vec{v}_3 \cdot \vec{w}_1) \vec{w}_1 + (\vec{v}_3 \cdot \vec{w}_2) \vec{w}_2$$

$$= \frac{1}{2}(1+2+3) \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} + \frac{1}{2}(1-2+3) \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \frac{3}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ -1 \end{bmatrix}$$

d) If we let $\mathbf{B} = \begin{bmatrix} \uparrow & \uparrow \\ \mathbf{w}_1 & \mathbf{w}_2 \\ \downarrow & \downarrow \end{bmatrix}$ where $\{\mathbf{w}_1, \mathbf{w}_2\}$ is the orthonormal basis found in part b,

what are the values of $\det(\mathbf{B}\mathbf{B}^T)$ and $\det(\mathbf{B}^T\mathbf{B})$?

[You don't need to know what \mathbf{w}_1 and \mathbf{w}_2 are to calculate these two numbers.]

$\mathbf{B}\mathbf{B}^T$ is MATRIX of ORTH. PROJ onto $\text{Span}\{\vec{w}_1, \vec{w}_2\}$, hence $\det(\mathbf{B}\mathbf{B}^T) = 0$

$\mathbf{B}^T\mathbf{B} = \begin{bmatrix} \leftarrow \vec{w}_1 \rightarrow \\ \leftarrow \vec{w}_2 \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow \\ \vec{w}_1 \\ \downarrow \\ \uparrow \\ \vec{w}_2 \\ \downarrow \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$ since $\{\vec{w}_1, \vec{w}_2\}$ is an ON basis.

Therefore $\det(\mathbf{B}^T\mathbf{B}) = \det(\mathbf{I}) = 1$.