Mathematics S-21b – Summer 2003 – Practice Exam #1 Solutions

(1) True or False. (Circle one) You need not give your reasoning.

(a) Consider a system $\mathbf{Ax} = \mathbf{b}$. This system is consistent if and only if rank($\mathbf{A}$) = rank([$\mathbf{A}|\mathbf{b}$]), where [$\mathbf{A}|\mathbf{b}$] denotes the augmented matrix.

TRUE – We know that a linear system is inconsistent if any row in the reduced row echelon form of the augmented matrix consists of all 0's except for a "1" in the rightmost column. This is the only way in which rank($\mathbf{A}$) and rank([$\mathbf{A}|\mathbf{b}$]) will differ.

(b) If $\mathbf{A}$ and $\mathbf{B}$ are n x n matrices such that the kernel of $\mathbf{A}$ is contained in the image of $\mathbf{B}$, then the matrix $\mathbf{AB}$ cannot be invertible.

FALSE - As a counterexample, choose both $\mathbf{A}$ and $\mathbf{B}$ to be invertible matrices. The kernel of $\mathbf{A}$ will be $\{\mathbf{0}\}$, the image of $\mathbf{B}$ will be all of $\mathbf{R}^n$ (which contains $\mathbf{0}$), and the product $\mathbf{AB}$ will be invertible.

(c) If $\mathbf{A}$ is an 8×5 matrix, then the kernel of $\mathbf{A}$ is at least three-dimensional.

FALSE – Such a matrix will represent a linear function from $\mathbf{R}^5$ to $\mathbf{R}^8$. If this function has rank 5, i.e. if its column vectors are linearly independent, then its kernel will consist only of the zero vector in $\mathbf{R}^5$.

(d) Consider the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in $\mathbf{R}^m$. Let $\mathbf{A}$ be a $p \times m$ matrix. If the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly dependent, then so are the vectors $\mathbf{Av}_1, \mathbf{Av}_2, \ldots, \mathbf{Av}_n$.

TRUE – If vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ are linearly dependent, then there are constants $c_1, \ldots, c_n$, not all zero, such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$. Apply (multiply by) the matrix $\mathbf{A}$ on both sides to get:
$\mathbf{A}(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n) = c_1\mathbf{Av}_1 + c_2\mathbf{Av}_2 + \cdots + c_n\mathbf{Av}_n = \mathbf{0}$. Thus, there's a nontrivial linear combination of the vectors $\{\mathbf{Av}_1, \mathbf{Av}_2, \ldots, \mathbf{Av}_n\}$ that sums to the zero vector, so they are also a linearly dependent set of vectors.

(e) rank($\mathbf{A}^2$) $\le$ rank($\mathbf{A}$) for any square matrix $\mathbf{A}$.

TRUE – Use the fact that $\text{im}(\mathbf{A}^2) \subseteq \text{im}(\mathbf{A})$ to conclude that $\dim(\text{im}(\mathbf{A}^2)) \le \dim(\text{im}(\mathbf{A}))$.

(f) There is a 4 x 4 matrix $\mathbf{A}$ such that image($\mathbf{A}$) and kernel($\mathbf{A}$) are the same subspace of $\mathbf{R}^4$.

TRUE - For example, define a transformation as follows: $T(\mathbf{e}_1) = \mathbf{e}_3$, $T(\mathbf{e}_2) = \mathbf{e}_4$, $T(\mathbf{e}_3) = \mathbf{0}$, $T(\mathbf{e}_4) = \mathbf{0}$. For the matrix $\mathbf{A}$ of this transformation, image($\mathbf{A}$) = kernel($\mathbf{A}$) = span$\{\mathbf{e}_3, \mathbf{e}_4\}$.

(2) Let $\mathbf{A}$ be the 4 x 5 matrix $\mathbf{A} = \begin{bmatrix} 1 & -2 & 1 & 5 & 0 \\ 1 & -2 & -1 & 1 & 0 \\ 3 & -6 & -1 & 7 & -1 \\ -1 & 2 & 0 & -3 & 0 \end{bmatrix}$.

(a) Find a basis for the kernel of $\mathbf{A}$ and its dimension.

Solution: We must solve the equation $\mathbf{Ax} = \mathbf{0}$. This is just a homogeneous system of 4 equations in 5 unknowns and is best solved using row reduction.

$$\left[\begin{array}{ccccc|c} 1 & -2 & 1 & 5 & 0 & 0 \\ 1 & -2 & -1 & 1 & 0 & 0 \\ 3 & -6 & -1 & 7 & -1 & 0 \\ -1 & 2 & 0 & -3 & 0 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccccc|c} 1 & -2 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

There are "leading 1's in the 1st, 3rd, and 5th columns of the reduced augmented matrix, so we'll be able to solve for the corresponding variables and introduce parameters for the others. Thus, we can choose $x_2 = s$ and $x_4 = t$ and solve for the others to get:

$$\left\{\begin{array}{l} x_1 = 2s - 3t \\ x_2 = s \\ x_3 = -2t \\ x_4 = t \\ x_5 = 0 \end{array}\right\}, \text{ or in vector form, } \mathbf{x} = s\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}.$$

Thus the kernel is of dimension 2 and these two vectors form a basis for ker($\mathbf{A}$).

(b) Find a basis for the image of **A** and its dimension.

Solution: We've shown that a basis for im(**A**) can be chosen by taking only those column vectors of the original matrix **A** that led to a "leading 1" in rref(**A**). Thus, in this example we may choose the 1st, 3rd, and 5th column vectors of **A**, namely:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \end{bmatrix}, \text{ and } \mathbf{v}_5 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}.$$

Thus the dimension of the image, i.e. the rank of **A**, must be 3.

(c) Find all solutions of the equation **Ax** = **b**, where **b** = $\begin{bmatrix} 3 \\ 5 \\ 1 \\ -4 \end{bmatrix}$.

Solution: This is solved in much the same way as we did in finding the kernel, only the system is no longer homogeneous and thus the matrix must be augmented with other than a column of zeros. Specifically:

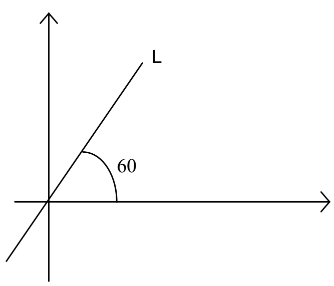
$$\left[\begin{array}{ccccc|c} 1 & -2 & 1 & 5 & 0 & 3 \\ 1 & -2 & -1 & 1 & 0 & 5 \\ 3 & -6 & -1 & 7 & -1 & 1 \\ -1 & 2 & 0 & -3 & 0 & -4 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccccc|c} 1 & -2 & 0 & 3 & 0 & 4 \\ 0 & 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 12 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So all solutions are of the form:

$$\left\{\begin{array}{l} x_1 = 4 + 2s - 3t \\ x_2 = s \\ x_3 = -1 - 2t \\ x_4 = t \\ x_5 = 12 \end{array}\right\} \text{ or in vector form } x = \begin{bmatrix} 4 \\ 0 \\ -1 \\ 0 \\ 12 \end{bmatrix} + s\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}.$$

Thus, the solution to the inhomogeneous system is just a translate of ker(**A**), the solution to the homogeneous system.

(3) Let T(**x**) = **Ax** be the linear transformation from $\mathbf{R}^2$ to $\mathbf{R}^2$ which first reflects a vector in the line L shown below and then dilates the resulting vector by the factor 3.



(a) Find the matrix **A**.

(b) Find $\mathbf{A}^4 \begin{bmatrix} -4 \\ 1 \end{bmatrix}$.

Solution: (a) There are several good ways to do this problem. Perhaps the simplest way is to construct the two columns of the matrix by finding T($\mathbf{e}_1$) and T($\mathbf{e}_2$).

Use the fact that $\text{Proj}_L(\mathbf{x}) = (\mathbf{x} \cdot \mathbf{u})\mathbf{u}$ where **u** is the unit vector $\mathbf{u} = \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}$ in the direction of the line L.

This gives $\text{Proj}_L(\mathbf{e}_1) = \frac{1}{2}\begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix} = \begin{bmatrix} 1/4 \\ \sqrt{3}/4 \end{bmatrix}$ and $\text{Proj}_L(\mathbf{e}_2) = \frac{\sqrt{3}}{2}\begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/4 \\ 3/4 \end{bmatrix}$.

From these we can compute $\text{Ref}_L(\mathbf{e}_1) = 2\,\text{Proj}_L(\mathbf{e}_1) - \mathbf{e}_1 = \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix}$, so $3\,\text{Ref}_L(\mathbf{e}_1) = \begin{bmatrix} -3/2 \\ 3\sqrt{3}/2 \end{bmatrix}$.

Likewise, we compute $\text{Ref}_L(\mathbf{e}_2) = 2\,\text{Proj}_L(\mathbf{e}_2) - \mathbf{e}_2 = \begin{bmatrix} \sqrt{3}/2 \\ 3/2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix}$, so $3\,\text{Ref}_L(\mathbf{e}_2) = \begin{bmatrix} 3\sqrt{3}/2 \\ 3/2 \end{bmatrix}$.

This gives us the matrix $\mathbf{A} = \begin{bmatrix} -3/2 & 3\sqrt{3}/2 \\ 3\sqrt{3}/2 & 3/2 \end{bmatrix} = \frac{3}{2}\begin{bmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$.

(b) Though you certainly can just carry out the matrix multiplications, it's much easier to simply note that this transformation simply flips and stretches by a factor of 3. If you do this four times, you end up in the same direction you started, only stretched by a factor of $3^4 = 81$. That is, $\mathbf{A}^4\begin{bmatrix} -4 \\ 1 \end{bmatrix} = 81\begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} -324 \\ 81 \end{bmatrix}$.

(4) A matrix can described geometrically by how it acts on any basis of vectors.

(a) Given the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$, show that these three vectors are linearly independent (and are therefore a basis for $\mathbf{R}^3$).

Solution: You can test for linear independence by making a matrix with these three vectors as its columns. The reduced row echelon form of this matrix is the 3×3 identity matrix, has rank 3, and its kernel is $\{\mathbf{0}\}$. Therefore, the three vectors are linearly independent.

(b) If we know that $\mathbf{A}\mathbf{v}_1 = \mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3$, $\mathbf{A}\mathbf{v}_2 = -2\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3$, and $\mathbf{A}\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2 - 3\mathbf{v}_3$, determine the matrix $\mathbf{A}$.

Solution: There are several good ways to approach this problem. Here's one:

Calculate each of the three vector sums using the vectors given above. This tells us that
$\mathbf{A}\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ 5 \end{bmatrix}$, $\mathbf{A}\begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -7 \\ -1 \\ -9 \end{bmatrix}$, and $\mathbf{A}\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -5 \\ -4 \\ -8 \end{bmatrix}$. These three can be rolled into one matrix statement, namely
$\mathbf{A}\begin{bmatrix} 2 & -1 & 2 \\ 1 & 4 & 3 \\ 3 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & -7 & -5 \\ -4 & -1 & -4 \\ 5 & -9 & -8 \end{bmatrix}$. Calling these two matrices $\mathbf{B}$ and $\mathbf{C}$, we can write this as $\mathbf{AB} = \mathbf{C}$. If we further note (from part (a)) that $\mathbf{B}$ is invertible, we can solve for $\mathbf{A}$ as $\mathbf{A} = \mathbf{CB}^{-1}$. After calculating this inverse matrix and doing the matrix multiplication, we get $\mathbf{A} = \begin{bmatrix} -98 & -47 & 83 \\ 13 & 6 & -12 \\ -108 & -52 & 91 \end{bmatrix}$.

Alternatively, you could find the matrix of this transformation relative to the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, then conjugate by the change of basis matrix to find $\mathbf{A}$, the matrix relative to the standard basis.