Math S-21b – Summer 2003 – Practice Exam #2

1) TRUE/FALSE

a) If T is a linear transformation from $\mathbf{R}^n$ to $\mathbf{R}^n$ which sends orthogonal vectors to orthogonal vectors, then T is an orthogonal transformation.	True or False
b) Let **A** be a square matrix with exactly one entry 1 in each row and in each column, the other entries being zero. Then **A** is an orthogonal matrix.	True or False
c) If **A** is an n × n matrix, then det(2**A**) = 2 (det **A**).	True or False
d) $\det(\mathbf{A}^T\mathbf{A}^2\mathbf{A}^T) = -16$ for some matrix **A** in $\mathbf{R}^{7\times 7}$, the space of 7×7 matrices with real entries. (Here $\mathbf{A}^T$ denotes the transpose of **A**.)	True or False
e) Let **A** be a 100 × 100 matrix with every entry equal to 1. Then det **A** = 1.	True or False

2) Consider the transformation $T: \mathbf{R}^{2\times 2} \to \mathbf{R}^{2\times 2}$ given by $T(\mathbf{A}) = \mathbf{A}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\mathbf{A}$.

a) Verify that T is a linear transformation.

b) Calculate $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$ and find the matrix of T with respect to the standard basis $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ of $\mathbf{R}^{2\times 2}$.

c) Find bases of the kernel and image of T.

d) Is T invertible? Explain.

3) Let V be the subspace of $\mathbf{R}^4$ spanned by the two vectors $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

a) Find an orthonormal basis $\{\mathbf{w}_1, \mathbf{w}_2\}$ for V using the Gram-Schmidt method.

b) Find the area of the parallelogram spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$.

c) Find the matrix **A** for orthogonal projection onto the subspace V.

d) Find the matrix **R** for reflection through the subspace V.

e) If **x** is the vector $\begin{bmatrix} 2 \\ 1 \\ 3 \\ -2 \end{bmatrix}$, find the vectors $\text{Proj}_V(\mathbf{x})$ and $\text{Ref}_V(\mathbf{x})$.

f) Find an orthonormal basis $\{\mathbf{w}_3, \mathbf{w}_4\}$ for the orthogonal complement $V^\perp$.

4) Find the quadratic function $f(x) = a + bx + cx^2$ that best fits the points (-1, 0), (0,0), (1,1), (2,0) in the sense of least squares.

5) We are given three vectors in $\mathbf{R}^4$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 4 \\ 0 \\ 0 \end{bmatrix}$.

a) Find the length of $\mathbf{v}_1$.

b) Find the area of the parallelogram determined by the vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$.

c) Find the 3-volume of the parallelepiped determined by the three vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.