Math S-21b – Summer 2003 – Practice Exam #2 Solutions

1) TRUE/FALSE

a) If T is a linear transformation from $\mathbf{R}^n$ to $\mathbf{R}^n$ which sends orthogonal vectors to orthogonal vectors, then T is an orthogonal transformation.

FALSE – The transformation might preserve angles but fail to preserve length (norm). An orthogonal transformation must preserve norm.

b) Let **A** be a square matrix with exactly one entry 1 in each row and in each column, the other entries being zero. Then **A** is an orthogonal matrix.

TRUE – This should be fairly clear. The columns represent the images of the standard basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ and the columns described above will clearly be mutually orthogonal unit vectors. This means that $\mathbf{A}^T\mathbf{A} = \mathbf{I}$, and $\mathbf{A}^T = \mathbf{A}^{-1}$, so the matrix is an orthogonal matrix.

c) If **A** is an $n \times n$ matrix, then $\det(2\mathbf{A}) = 2(\det \mathbf{A})$.

FALSE – The multilinearity property of the determinant says that the determinant of a matrix will be linear in any single row or column, but if you scale the entire matrix by a constant, every row (or column) will be scaled by that constant, so the net result will be that $\det(k\mathbf{A}) = k^n(\det \mathbf{A})$.

d) $\det(\mathbf{A}^T\mathbf{A}^2\mathbf{A}^T) = -16$ for some matrix **A** in $\mathbf{R}^{7\times7}$, the space of 7×7 matrices with real entries. (Here $\mathbf{A}^T$ denotes the transpose of **A**.)

FALSE – Use the properties that for square matrices, $\det(\mathbf{A}^T) = \det(\mathbf{A})$ and $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$. This gives us that $\det(\mathbf{A}^T\mathbf{A}^2\mathbf{A}^T) = \det(\mathbf{A}^T)\det(\mathbf{A}^2)\det(\mathbf{A}^T) = \det(\mathbf{A})\det(\mathbf{A})\det(\mathbf{A})\det(\mathbf{A}) = [\det(\mathbf{A})]^4 = -16$ and there are no real numbers a such that $a^4 = -16$, hence there are no such matrices with this property.

e) Let **A** be a 100×100 matrix with every entry equal to 1. Then $\det \mathbf{A} = 1$.

FALSE – Clearly all rows are identical and if any two rows are identical the determinant will be 0.

2) Consider the transformation $T: \mathbf{R}^{2\times2} \to \mathbf{R}^{2\times2}$ given by $T(\mathbf{A}) = \mathbf{A}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\mathbf{A}$.

a) Verify that T is a linear transformation.

Solution: This is a formal verification. Call $\mathbf{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, so $T(\mathbf{A}) = \mathbf{AJ} - \mathbf{JA}$. We calculate:

$T(\alpha\mathbf{A} + \beta\mathbf{B}) = (\alpha\mathbf{A} + \beta\mathbf{B})\mathbf{J} - \mathbf{J}(\alpha\mathbf{A} + \beta\mathbf{B}) = \alpha\mathbf{AJ} + \beta\mathbf{BJ} - \alpha\mathbf{JA} - \beta\mathbf{JB} = \alpha(\mathbf{AJ} - \mathbf{JA}) + \beta(\mathbf{BJ} - \mathbf{JB}) = \alpha T(\mathbf{A}) + \beta T(\mathbf{B})$, so T is a linear transformation.

b) Calculate $T(\begin{bmatrix} a & b \\ c & d \end{bmatrix})$ and find the matrix of T with respect to the standard basis $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ of $\mathbf{R}^{2\times2}$.

Solution: Just do the calculation directly and you'll get $T(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{bmatrix} b+c & d-a \\ d-a & -(b+c) \end{bmatrix}$.

Thus, in terms of the given basis, a matrix with coordinates $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ will be transformed into one with coordinates

$\begin{bmatrix} b+c \\ d-a \\ d-a \\ -(b+c) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$. Therefore the matrix of T will be $\begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{bmatrix}$..

2c) Find bases of the kernel and image of T.

Solution: From the above calculation, we can see that $T(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = (b+c)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + (d-a)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, so the matrices $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ are a basis for the image.

For the kernel, set $T(\mathbf{A}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. This implies that $b + c = 0$ and $d - a = 0$, or $c = -b$ and $d = a$. Thus any matrix in the kernel will be of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, so the matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ form a basis for the kernel of this linear transformation. You can also work directly with the 4×4 matrix found above to determine the image and kernel of this transformation.

d) Is T invertible? Explain.
Solution: No. It's matrix is clearly not invertible and the kernel is not trivial.

3) Let V be the subspace of $\mathbf{R}^4$ spanned by the two vectors $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

a) Find an orthonormal basis $\{\mathbf{w}_1, \mathbf{w}_2\}$ for V using the Gram-Schmidt method.
Solution: $\mathbf{v}_1$ is already a unit vector, so there's no need to normalize, i.e. $\mathbf{w}_1 = \mathbf{v}_1$. If you subtract the projection of $\mathbf{v}_2$ in the direction of $\mathbf{w}_1$ and normalize it, you get the vector $\mathbf{w}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

b) Find the area of the parallelogram spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$.

Solution: Let $\mathbf{A} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then area$\{\mathbf{v}_1, \mathbf{v}_2\} = \sqrt{\det(\mathbf{A}^T\mathbf{A})} = \sqrt{\det\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}} = \sqrt{2}$.

c) Find the matrix **A** for orthogonal projection onto the subspace V.

Solution: Using the orthonormal basis derived in part a), let $\mathbf{B} = \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1/\sqrt{2} \\ 1 & 0 \\ 0 & 0 \\ 0 & 1/\sqrt{2} \end{bmatrix}$. Then the matrix for orthogonal projection onto the subspace V spanned by $\{\mathbf{v}_1, \mathbf{v}_2\}$ (or equivalently by $\{\mathbf{w}_1, \mathbf{w}_2\}$) will be given by $\mathbf{B}\mathbf{B}^T$.

That is, $\text{Proj}_V = \begin{bmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{bmatrix}$.

3d) Find the matrix **R** for reflection through the subspace V.

Solution: We learned that $\text{Ref}_V = 2\text{Proj}_V - \text{Identity}$, or in terms of matrices, the matrix of reflection through the subspace V will be $2\mathbf{B}\mathbf{B}^T - \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

e) If **x** is the vector $\begin{bmatrix} 2 \\ 1 \\ 3 \\ -2 \end{bmatrix}$, find the vectors $\text{Proj}_V(\mathbf{x})$ and $\text{Ref}_V(\mathbf{x})$.

Solution: $\text{Proj}_V(\mathbf{x}) = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\text{Ref}_V(\mathbf{x}) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \\ 2 \end{bmatrix}$.

f) Find an orthonormal basis $\{\mathbf{w}_3, \mathbf{w}_4\}$ for the orthogonal complement $V^\perp$.

Solution: Start with the fact that $V^\perp = (\text{im } \mathbf{A})^\perp = \ker(\mathbf{A}^T)$. This gives us

$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right]$ from which we find the solutions $\mathbf{x} = s\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. Fortunately, these two basis vectors are orthogonal, so all we have to do is normalize to get $\mathbf{w}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{w}_4 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

4) Find the quadratic function $f(x) = a + bx + cx^2$ that best fits the points $(-1, 0)$, $(0,0)$, $(1,1)$, $(2,0)$ in the sense of least squares.

Solution: As with all of the data-fitting examples you've seen, you start by trying to solve the impossible case of all the data fitting perfectly. In this case, this translates into the four equations $\begin{Bmatrix} a - b + c = 0 \\ a = 0 \\ a + b + c = 1 \\ a + 2b + 4c = 0 \end{Bmatrix}$. In matrix form, this gives us $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ or $\mathbf{Ac} = \mathbf{b}$. The normal equation for a least squares solution is $\mathbf{A}^T\mathbf{Ac} = \mathbf{A}^T\mathbf{b}$.

That is, $\begin{bmatrix} 4 & 2 & 6 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Solve this by row reduction or matrix inversion to get $a = \frac{9}{20}, b = \frac{7}{20}, c = -\frac{1}{4}$.

5) We are given three vectors in $\mathbf{R}^4$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 4 \\ 0 \\ 0 \end{bmatrix}$.

a) Find the length of

Solution: $\|\mathbf{v}_1\| = \sqrt{1+4+4} = \sqrt{9} = 3$.

b) Find the area of the parallelogram determined by the vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$.

Solution: Let $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}$. Area = $\sqrt{\det(\mathbf{A}^T\mathbf{A})} = \sqrt{\det\begin{bmatrix} 9 & 1 \\ 1 & 4 \end{bmatrix}} = \sqrt{35}$

c) Find the 3-volume of the parallelepiped determined by the three vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Solution: Let $\mathbf{A} = \begin{bmatrix} 1 & 1 & 3 \\ -2 & 1 & 4 \\ 0 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix}$.

Volume = $\sqrt{\det(\mathbf{A}^T\mathbf{A})} = \sqrt{\det\begin{bmatrix} 9 & 1 & -5 \\ 1 & 4 & 7 \\ -5 & 7 & 25 \end{bmatrix}} == \sqrt{459-60-135} = \sqrt{264} = 2\sqrt{66}$