# Math S-21b Practice Final Exam – Summer 2002

1) **True/False**

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|---|---|
| a) If $\lambda$ is an eigenvalue of **A** with eigenvector **v**, $\mu$ is a distinct eigenvalue with eigenvector **w**, then **v** + **w** is also an eigenvector of **A**. | TRUE or FALSE |
| b) If **v** is an eigenvector of both **A** and **B**, then **v** is an eigenvector of **AB**. | TRUE or FALSE |
| c) If **A** is a real $2 \times 2$ matrix and $\det(\mathbf{A}) < 0$, then all eigenvalues of **A** are real. | TRUE or FALSE |
| d) If **A** is a real $4 \times 4$ matrix with $\mathbf{A}^2 = \mathbf{A}$, then the only possible eigenvalues of **A** are 0 and 1. | TRUE or FALSE |
| e) If the eigenvalues for a $2 \times 2$ matrix are $\frac{11}{12}$ and $\frac{13}{12}$, and **v** is neither **0** nor an eigenvector for **A**, then $\left\|\mathbf{A}^n\mathbf{v}\right\|$ will go to $\infty$ as $n \to \infty$. | TRUE or FALSE |
| f) Suppose that **A** and **B** are simultaneously diagonalizable $n \times n$ matrices, i.e. there is an invertible matrix **S** such that $\mathbf{S}^{-1}\mathbf{AS}$ and $\mathbf{S}^{-1}\mathbf{BS}$ are both diagonal. Then **AB** must equal **BA**. | TRUE or FALSE |
| g) If there are orthonormal eigenbases of $\mathbf{R}^n$ for each of two matrices **A** and **B**, then there is also an orthonormal eigenbasis for **A** + **B**. | TRUE or FALSE |

2) Consider the matrix $\mathbf{A} = \begin{bmatrix} -1 & 0 & 1 & 0 & 3 \\ 1 & 0 & 3 & 4 & 1 \\ 1 & 0 & 1 & 2 & -1 \end{bmatrix}$.

a) Find a basis for ker **A**, the kernel of **A**.
b) Find a basis for im **A**, the image of **A.**
c) Using the Gram-Schmidt process on your answer to (b), find an orthonormal basis for im **A**.
d) Find the matrix for orthogonal projection onto the subspace im **A**.
e) Find the orthogonal projection of the standard basis vector $\mathbf{j} = \mathbf{e}_2$ in the subspace im **A**.

3) Let T: $\mathbf{R}^3 \to \mathbf{R}^3$ be orthogonal projection onto the *z*-axis. Find the matrix M of this linear transformation, with respect to the orthonormal basis:

$$\mathbf{v}_1 = \tfrac{1}{7}\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix},\ \mathbf{v}_2 = \tfrac{1}{7}\begin{bmatrix} 6 \\ 2 \\ -3 \end{bmatrix},\ \mathbf{v}_3 = \tfrac{1}{7}\begin{bmatrix} -3 \\ 6 \\ -2 \end{bmatrix}.$$

4) Let $\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

a) Find all real/complex eigenvalues of A with their algebraic multiplicities.
b) Does A have a real/complex eigenbasis? If so, find one.
c) Is A diagonalizable? Why or why not?
d) Let T: $\mathbf{R}^4 \to \mathbf{R}^4$ be the linear transformation given by $T(\mathbf{v}) = \mathbf{Av}$. Describe T geometrically.

5) Let $q(x_1, x_2) = -7x_1^2 + 48x_1x_2 + 7x_2^2 = \mathbf{x}^T\mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a symmetric $2 \times 2$ matrix.

a) Does $\mathbf{R}^2$ have an orthonormal eigenbasis with respect to $\mathbf{A}$? If so, find one. If not, why not?
b) Sketch the curve $q(x_1, x_2) = 25$.

6) Consider the system $\left\{ \begin{array}{l} \dfrac{dx}{dt} = y^2 - x \\ \dfrac{dy}{dt} = x^2 - y \end{array} \right\}$.

a) Perform the qualitative phase plane analysis for this system (i.e., find the null clines, equilibrium points, and general directions). Carry this out in the whole $xy$-plane (not just the first quadrant).

b) List the equilibrium points of the system above, and determine their stability. That is, linearize the system at each equilibrium and do the eigenvalue-eigenvector analysis. Give a rough sketch of some solutions, particularly in the vicinity of the equilibria.

7) a) Find all solutions of the differential equation

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} - 6x = 6t$$

b) Find a solution which satisfies $x'(0) = x''(0) = 30$. Is this solution unique?

**Note**: Neither the length of this practice exam nor the topics covered are necessarily indicative of what you should expect on the actual final exam. They are simply a subset of the problems that were on last summer's Final Exam