SOLUTIONS TO PRACTICE FINAL EXAM – SUMMER 2003 – MATH S-21b

1. $\boxed{T/F}$ (a) FALSE $A\vec{v} = \lambda\vec{v}$, $A\vec{w} = \mu\vec{w}$, $A(\vec{v}+\vec{w}) = A\vec{v} + A\vec{w} = \lambda\vec{v} + \mu\vec{w}$

   $\lambda\vec{v} + \mu\vec{w}$ not a multiple of $\vec{v}+\vec{w}$ unless $\lambda = \mu$.

   (b) TRUE $A\vec{v} = \lambda\vec{v}$, $B\vec{v} = \mu\vec{v} \Rightarrow AB\vec{v} = A(\mu\vec{v}) = \mu A\vec{v} = \mu\lambda\vec{v}$

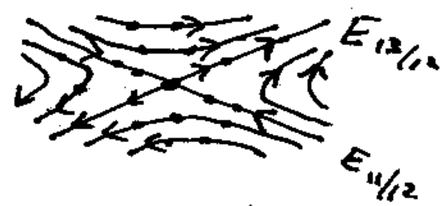
   (c) TRUE If A had complex e-vals $\lambda, \bar{\lambda}$, then $\det A = \lambda\bar{\lambda} = |\lambda|^2 > 0$, so A cannot have complex e-vals.

   (d) TRUE If $A^2 = A$, then for any e-vector $\vec{v}$ we'll have $A\vec{v} = \lambda\vec{v}$ for any e-value λ and $A^2\vec{v} = AA\vec{v} = A(\lambda\vec{v}) = \lambda A\vec{v} = \lambda^2\vec{v}$

   $\therefore A\vec{v} = \lambda\vec{v} \Rightarrow \lambda^2\vec{v} = \lambda\vec{v}$

   So $(\lambda^2 - \lambda)\vec{v} = \vec{0} \Rightarrow \lambda^2 - \lambda = \lambda(\lambda - 1) = 0 \Rightarrow \lambda = 0$ or $\lambda = 1$.

   (e) TRUE PICTURE SAYS IT ALL.

   only trajectories that don't tend toward ∞ are those where we start on $E_{11/12}$, i.e. an eigenvector w/ $\lambda = 11/12$.

   (f) TRUE $S^{-1}AS = D_1$, $S^{-1}BS = D_2$ DIAGONAL MATRICES.

   $A = SD_1S^{-1}$ $B = SD_2S^{-1} \Rightarrow AB = SD_1S^{-1}SD_2S^{-1} = SD_1D_2S^{-1}$
   $= SD_2D_1S^{-1}$ (since diag. matrices commute)
   $= SD_2S^{-1}SD_1S^{-1} = BA$ $\therefore AB = BA$

   (g) TRUE If both A and B have ON eigenbases, then by the SPECTRAL THEOREM, both A and B must be symmetric. But then $(A+B)^T = A^T + B^T = A + B \Rightarrow A+B$ is symmetric. Therefore $A+B$ must be orthogonally diagonalizable, so it also has an orthonormal eigenbasis.

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2. (a) $$\text{rref}(A) = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \quad\;\; \uparrow \qquad\quad \uparrow \quad \uparrow \\ \quad\;\; r \qquad\quad s \quad t \end{array} \Rightarrow \begin{aligned} x_1 &= -s+2t \\ x_2 &= r \\ x_3 &= -s-t \\ x_4 &= s \\ x_5 &= t \end{aligned} \Rightarrow \vec{x} = r\begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix} + s\begin{bmatrix}-1\\0\\-1\\1\\0\end{bmatrix} + t\begin{bmatrix}2\\0\\-1\\0\\1\end{bmatrix}$$

   basis for Ker A.

   (b) $$\text{Im}(A) = \text{Span}\left\{\begin{bmatrix}-1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\3\\1\end{bmatrix}\right\} = \text{Span}\{\vec{v}_1, \vec{v}_2\} = V$$

   (c) $$\frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{3}}\begin{bmatrix}-1\\1\\1\end{bmatrix} = \vec{w}_1 \qquad \vec{v}_2 - \text{Proj}_{\vec{w}_1}(\vec{v}_2) = \begin{bmatrix}1\\3\\1\end{bmatrix} - \frac{1}{3}(3)\begin{bmatrix}-1\\1\\1\end{bmatrix} = \begin{bmatrix}2\\2\\0\end{bmatrix}$$

   NORMALIZE $\Rightarrow \vec{w}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}$

   (d) WRITE $$B = [\vec{w}_1, \vec{w}_2] = \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{3} & 0 \end{bmatrix} \qquad \text{Proj}_V = BB^T = \frac{1}{6}\begin{bmatrix} 5 & 1 & -2 \\ 1 & 5 & 2 \\ -2 & 2 & 2 \end{bmatrix}$$

   (e) $$\text{Proj}_V(\vec{e}_2) = \frac{1}{6}\begin{bmatrix}1\\5\\2\end{bmatrix}.$$

(3) $S = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 2 & 6 & -3 \\ 3 & 2 & 6 \\ 6 & -3 & -2 \end{bmatrix}$ will be an orthogonal MATRIX, so $S^T = S^{-1}$.

MATRIX of ORTH. PROJ. onto z-AXIS is $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

So $[A]_{\mathcal{B}} = S^{-1}AS = \frac{1}{49}\begin{bmatrix} 2 & 3 & 6 \\ 6 & 2 & -3 \\ -3 & 6 & -2 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 6 & -3 \\ 3 & 2 & 6 \\ 6 & -3 & -2 \end{bmatrix} = \frac{1}{49}\begin{bmatrix} 36 & -18 & -12 \\ -18 & 9 & 6 \\ -12 & 6 & 4 \end{bmatrix}$

is MATRIX rel. to basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

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(4) $\lambda I - A = \begin{bmatrix} \lambda & 0 & 0 & -1 \\ 0 & \lambda & -1 & 0 \\ 0 & -1 & \lambda & 0 \\ -1 & 0 & 0 & \lambda \end{bmatrix}$ $\quad P_A(\lambda) = \lambda \begin{vmatrix} \lambda & -1 & 0 \\ -1 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix} + 1 \begin{vmatrix} 0 & \lambda & -1 \\ 0 & -1 & \lambda \\ -1 & 0 & 0 \end{vmatrix}$

$= \lambda[\lambda(\lambda^2-1)] + 1[(-1)(\lambda^2-1)]$

$= (\lambda^2-1)(\lambda^2-1) = (\lambda-1)^2(\lambda+1)^2 = 0$

e-values are $\overset{\lambda_2 =}{\lambda_1} = 1$ (Alg. mult $= 2$) and $\overset{\lambda_3 =}{\lambda_4} = -1$ (Alg mult $= 2$).

$\lambda_2 = \lambda_1 = 1 \Rightarrow \vec{x} = s\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$; $\lambda_3 = \lambda_4 = -1 \Rightarrow \vec{x} = s\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$\vec{v}_1 \quad \vec{v}_2 \qquad\qquad \vec{v}_3 \quad \vec{v}_4$.

GM = 2 $\qquad\qquad$ GM = 2

(b) $\Rightarrow$ REAL EIGENBASIS $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$

Spectrum $\{1, 1, -1, -1\}$.

(c) A IS DIAGONALIZABLE since there is an eigenbasis. Also, since A is symmetric, it has an ON eigenbasis.

(d) $\left.\begin{matrix} T(\vec{v}_1) = \vec{v}_1 \\ T(\vec{v}_2) = \vec{v}_2 \end{matrix}\right\}$ $V = \text{Span}\{\vec{v}_1, \vec{v}_2\}$ $\qquad$ T is Ref_V

$\left.\begin{matrix} T(\vec{v}_3) = -\vec{v}_3 \\ T(\vec{v}_4) = -\vec{v}_4 \end{matrix}\right\}$ $V^\perp = \text{Span}\{\vec{v}_3, \vec{v}_4\}$ $\qquad$ (reflection across subspace V).

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(5) $q(\vec{x}) = \vec{x}^T A \vec{x}$ for $A = \begin{bmatrix} -7 & 24 \\ 24 & -7 \end{bmatrix}$. A is symmetric, hence it has an ON eigenbasis.

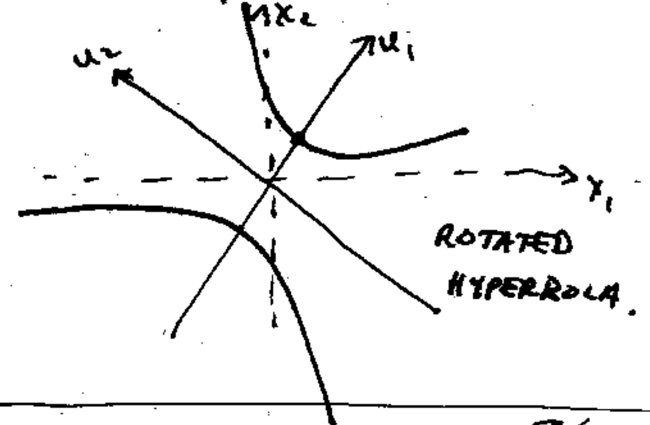
$\lambda I - A = \begin{bmatrix} \lambda+7 & -24 \\ -24 & \lambda-7 \end{bmatrix} \Rightarrow P_A(\lambda) = \lambda^2 - 625 = (\lambda-25)(\lambda+25) = 0$

$\Rightarrow$ eigenvalues are $\lambda_1 = 25$, $\lambda_2 = -25$ $\Rightarrow$ ON e-vectors $\vec{v}_1 = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix} \Rightarrow$

(5 continued) Change of basis $S = \frac{1}{5}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$ (ORTHOGONAL) $\quad S^TAS = D = \begin{bmatrix} 25 & 0 \\ 0 & -25 \end{bmatrix}$

$A = SDS^T$

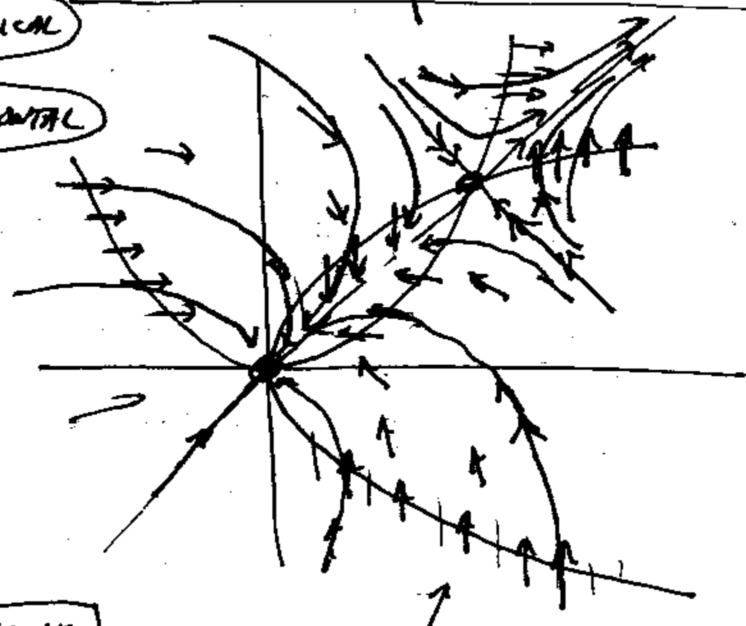
$$f(\vec{x}) = \vec{x}^T A \vec{x} = \vec{x}^T S D S^T \vec{x}$$
$$= (S^{-1}\vec{x}) D (S^{-1}\vec{x}) = \vec{u} D \vec{u} = 25u_1^2 - 25u_2^2 = 25$$
$$\Rightarrow \boxed{u_1^2 - u_2^2 = 1}$$



(6) $\begin{cases} \frac{dx}{dt} = y^2 - x = f(x,y) \\ \frac{dy}{dt} = x^2 - y = g(x,y) \end{cases}$.

NULLCLINES $\frac{dx}{dt} = 0 \Rightarrow x = y^2$ VERTICAL

$\frac{dy}{dt} = 0 \Rightarrow y = x^2$ HORIZONTAL

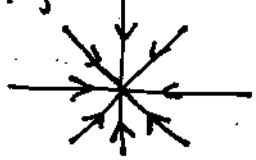


EQUILIBRIA AT $(0,0)$ and $(1,1)$

LINEARIZATION AT $(0,0)$:

$$J_F = \begin{bmatrix} -1 & 2y \\ 2x & -1 \end{bmatrix}$$

$$J_F(0,0) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{DIAGONAL}$$



$\boxed{\text{SINK}}$ STABLE

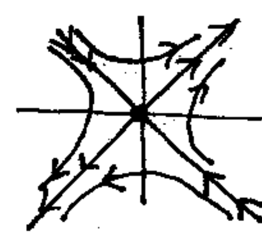
LINEARIZATION AT $(1,1)$:

$$J_F(1,1) = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} \quad \begin{bmatrix} \lambda+1 & -2 \\ -2 & \lambda+1 \end{bmatrix} \Rightarrow \lambda^2 + 2\lambda + 1 - 4$$
$$= \lambda^2 + 2\lambda - 3 = 0$$
$$(\lambda+3)(\lambda-1) = 0$$

e-vals $\lambda_1 = -3 \quad \lambda_2 = +1$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



unstable EQUILIBRIUM AT $(1,1)$.

7 (a) to solve $\frac{d^2x}{dt^2} - \frac{dx}{dt} - 6x = 6t$, first solve homogeneous eqn

or $\frac{d^2x}{dt^2} - \frac{dx}{dt} - 6x = 0$ for solution $x_h(t)$.

then find particular solution $x_p(t)$ for inhomogeneous eqn.

General solution will then be $x(t) = x_h(t) + x_p(t)$.

<u>Homog soln:</u> characteristic polynomial is $\lambda^2 - \lambda - 6 = 0$

$\Rightarrow (\lambda - 3)(\lambda + 2) = 0 \Rightarrow \lambda_1 = 3 \quad \lambda_2 = -2$.

$\Rightarrow$ solution $\{e^{3t}, e^{-2t}\}$.

So $\boxed{x_h(t) = c_1 e^{3t} + c_2 e^{-2t}}$ for arb. constants c_1, c_2.

<u>Inhomog. soln</u>: Try soln of form $x(t) = at^2 + bt + c$.

$\frac{dx}{dt} = 2at + b$

$\frac{d^2x}{dt^2} = 2a$

So $\frac{d^2x}{dt^2} - \frac{dx}{dt} - 6x = -6at^2 + (-2a - 6b)t + (2a - b - 6c) = 6t$

$\Rightarrow a = 0$, $-2a - 6b = 6 \Rightarrow b = -1$ $\quad 2a - b - 6c = 0 \Rightarrow c = +\frac{1}{6}$

$\Rightarrow \boxed{x_p(t) = -t + \frac{1}{6}}$ <u>Check:</u> $x_p'' - x_p' - 6x_p = 0 + 1 - 6(-t + \frac{1}{6})$

$= 1 + 6t - 1 = 6t$ ✓.

So General SOLUTION is $\boxed{x(t) = c_1 e^{3t} + c_2 e^{-2t} - t + \frac{1}{6}}$

(b) $x'(t) = 3c_1 e^{3t} - 2c_2 e^{-2t} - 1$ $\qquad x'(0) = 3c_1 - 2c_2 - 1 = 30$

$x''(t) = 9c_1 e^{3t} + 4c_2 e^{-2t}$ $\qquad x''(0) = 9c_1 + 4c_2 = 30$

$$\Rightarrow \begin{matrix} 3c_1 - 2c_2 = 31 \\ 9c_1 + 4c_2 = 30 \end{matrix} \Rightarrow \begin{bmatrix} 3 & -2 \\ 9 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 31 \\ 30 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{30} \begin{bmatrix} 4 & 2 \\ -9 & 3 \end{bmatrix} \begin{bmatrix} 31 \\ 30 \end{bmatrix} = \frac{1}{30} \begin{bmatrix} 184 \\ -189 \end{bmatrix} = \begin{bmatrix} 92/15 \\ -63/10 \end{bmatrix}$$

$$\Rightarrow \boxed{x(t) = \frac{92}{15} e^{3t} - \frac{63}{10} e^{-2t} - t + \frac{1}{6}}$$