1.3

1. a. No solution, since the last row indicates $0 = 1$.

   b. The unique solution is $x = 5,\ y = 6$.

   c. Infinitely many solutions; the first variable can be chosen freely.

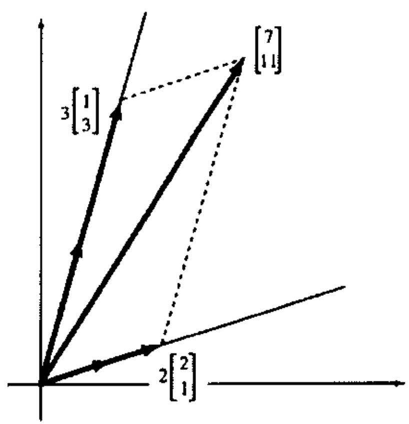
2. The rank is 3 since each row contains a leading one.

3. This matrix has rank 1 since its rref is $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

4. This matrix has rank 2 since its rref is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

5. a. $x \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$

   b. The solution of the system in part (a) is $x = 3,\ y = 2$.



24. By Fact 1.3.4, rref $(A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Using Fact 1.3.4 again, we can conclude that the system $A\vec{x} = \vec{c}$ has a unique solution as well.

25. In this case, rref(A) has a row of zeros, so that rank$(A) < 4$; there will be a nonleading variable. The system $A\vec{x} = \vec{c}$ could have infinitely many solutions (for example, when $\vec{c} = \vec{0}$) or no solutions (for example, when $\vec{c} = \vec{b}$), but it cannot have a unique solution, by Fact 1.3.4.

26. From Exercise 22 we know that rref $(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Since all variables are leading, the system $A\vec{x} = \vec{c}$ cannot have infinitely many solutions, but it could have a unique solution (for example, if $\vec{c} = \vec{b}$) or no solutions at all (compare with Example 13).

27. a. True, since rref$[A \vdots \vec{b}]$ contains a row $[0\ 0\ \ldots\ 0 \vdots\ 1]$.

b. False; as a counterexample, consider the case when A is the zero matrix and $\vec{b}$ the zero vector.

47. a. $\vec{x} = \vec{0}$ is a solution.

b. This holds by part (a) and Fact 1.3.3.

c. If $\vec{x}_1$ and $\vec{x}_2$ are solutions, then $A\vec{x}_1 = \vec{0}$ and $A\vec{x}_2 = \vec{0}$.

Therefore, $A(\vec{x}_1+\vec{x}_2) = A\vec{x}_1 + A\vec{x}_2 = \vec{0}+\vec{0} = \vec{0}$, so that $\vec{x}_1+\vec{x}_2$ is a solution as well. Note that we have used Fact 1.3.7a.

d. $A(k\vec{x}) = k(A\vec{x}) = k\vec{0} = \vec{0}$

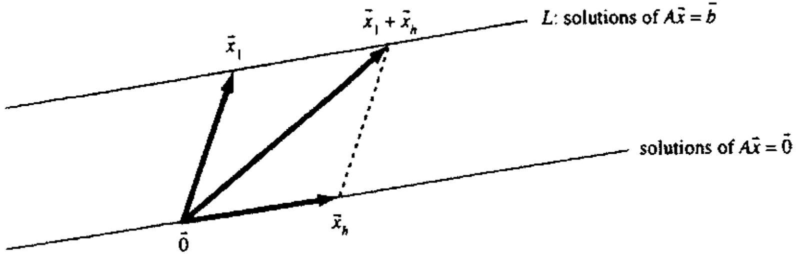
We have used Fact 1.3.7b.

48. The fact that $\vec{x}_1$ is a solution of $A\vec{x} = \vec{b}$ means that $A\vec{x}_1 = \vec{b}$.

a. $A(\vec{x}_1+\vec{x}_h) = A\vec{x}_1 + A\vec{x}_h = \vec{b}+\vec{0} = \vec{b}$

b. $A(\vec{x}_2-\vec{x}_1) = A\vec{x}_2 - A\vec{x}_1 = \vec{b}-\vec{b} = \vec{0}$

c. Parts (a) and (b) show that the solutions of $A\vec{x} = \vec{b}$ are exactly the vectors of the form $\vec{x}_1+\vec{x}_h$, where $\vec{x}_h$ is a solution of $A\vec{x} = \vec{0}$; indeed if $\vec{x}_2$ is a solution of $A\vec{x} = \vec{b}$, we can write $\vec{x}_2 = \vec{x}_1 + (\vec{x}_2 - \vec{x}_1)$, and $\vec{x}_2 - \vec{x}_1$ will be a solution of $A\vec{x} = \vec{0}$, by part (b).
Geometrically, the vectors of the form $\vec{x}_1+\vec{x}_h$ are those whose tips are on the line L sketched below; the line L runs through the tip of $\vec{x}_1$ and is parallel to the given line consisting of the solutions of $A\vec{x} = \vec{0}$.



56. We can use technology to determine that the system $\begin{bmatrix} 30 \\ -1 \\ 38 \\ 56 \\ 62 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 7 \\ 1 \\ 9 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 6 \\ 3 \\ 2 \\ 8 \end{bmatrix} + x_3 \begin{bmatrix} 9 \\ 2 \\ 3 \\ 5 \\ 2 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ -5 \\ 4 \\ 7 \\ 9 \end{bmatrix}$

is inconsistent; therefore, the vector $\begin{bmatrix} 30 \\ -1 \\ 38 \\ 56 \\ 62 \end{bmatrix}$ is not a linear combination of the other four vectors.