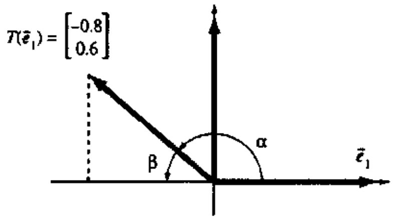
4. $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \sqrt{2}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

By Fact 2.2.3, this matrix represents a rotation-dilation: a rotation by 45° followed by a dilation by $\sqrt{2}$.

5.



Note that $\sin\beta = 0.6$ so that $\beta = \arcsin(0.6) \approx 0.6435$, and $\alpha = \pi - \beta \approx 2.498$.

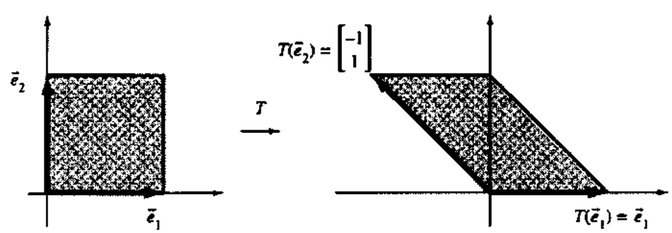
6. By Fact 2.2.5, $\text{proj}_L \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \left(\vec{u} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)\vec{u}$, where $\vec{u}$ is a unit vector on L. To get $\vec{u}$, we normalize $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$:

$\vec{u} = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, so that $\text{proj}_L \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{5}{3} \cdot \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{10}{9} \\ \frac{5}{9} \\ \frac{10}{9} \end{bmatrix}$.

7. By Fact 2.2.7, $\text{ref}_L \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 2\left(\vec{u} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)\vec{u} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, where $\vec{u}$ is a unit vector on L. To get $\vec{u}$, we normalize $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$:

$\vec{u} = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, so that $\text{ref}_L \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 2\frac{5}{3} \cdot \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{11}{9} \\ \frac{1}{9} \\ \frac{11}{9} \end{bmatrix}$.

8. Examine the effect this transformation has on the unit square (compare with Example 6):



This is a shear parallel to the $\vec{e}_1$ axis.

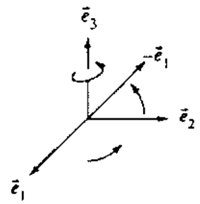
18. If T is indeed a reflection-dilation, then the factor of dilation is $\|T(\vec{e}_1)\| = \left\|\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right\| = 5$, since a reflection leaves the length of a vector unchanged.

Now write $\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} = 5\begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}$; we need to verify that the matrix $\begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}$ represents a reflection. The result of Exercise 13 shows that this is indeed the case, with $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$.

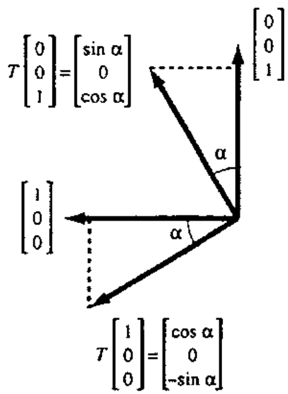
19. $T(\vec{e}_1) = \vec{e}_1$, $T(\vec{e}_2) = \vec{e}_2$, and $T(\vec{e}_3) = \vec{0}$, so that the matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

20. $T(\vec{e}_1) = \vec{e}_1$, $T(\vec{e}_2) = -\vec{e}_2$, and $T(\vec{e}_3) = \vec{e}_3$, so that the matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

21. $T(\vec{e}_1) = \vec{e}_2$, $T(\vec{e}_2) = -\vec{e}_1$, and $T(\vec{e}_3) = \vec{e}_3$, so that the matrix is $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

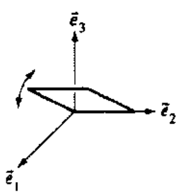


22. Sketch the $\vec{e}_1 - \vec{e}_3$ plane, as viewed from the positive $\vec{e}_2$ axis.



Since $T(\vec{e}_2) = \vec{e}_2$, the matrix is $\begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$.

23. $T(\vec{e}_1) = \vec{e}_3$, $T(\vec{e}_2) = \vec{e}_2$, and $T(\vec{e}_3) = \vec{e}_1$, so that the matrix is $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.



this is for Reflection across $x=z$ plane

For Reflection across $y=z$ plane, we have $T(\vec{e}_1) = \vec{e}_1$
$T(\vec{e}_2) = \vec{e}_3$
$T(\vec{e}_3) = \vec{e}_2$
$\Rightarrow A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

25. The matrix $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ represents a shear parallel to the $\vec{e}_1$ axis, and its inverse $A^{-1} = \begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix}$ represents such a shear as well, but "the other way."

26. In view of part a of Definition 2.2.4, we will first find the vectors $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ such that $T(\vec{v}) = \vec{v}$. We need to solve the system $\left| \begin{matrix} -v_1 + 4v_2 = v_1 \\ -v_1 + 3v_2 = v_2 \end{matrix} \right|$ or $\left| \begin{matrix} -2v_1 + 4v_2 = 0 \\ -v_1 + 2v_2 = 0 \end{matrix} \right|$, with solutions $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$, where t is an arbitrary real number.

To show that T is a shear parallel to the line L consisting of all vectors of the form $\begin{bmatrix} 2t \\ t \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, we will verify part b of Definition 2.2.4:

$T(\vec{x}) - \vec{x} = \begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 + 4x_2 \\ -x_1 + 3x_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_1 + 4x_2 \\ -x_1 + 2x_2 \end{bmatrix} = (-x_1 + 2x_2)\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, is indeed on L, for all $\vec{x}$ in $\mathbb{R}^2$.

27. Yes; let us verify the two requirements expressed in Fact 2.2.1. Write $F(\vec{x}) = L(T(\vec{x}))$.

a. $F(\vec{v} + \vec{w}) = L(T(\vec{v} + \vec{w})) = L(T(\vec{v}) + T(\vec{w})) = L(T(\vec{v})) + L(T(\vec{w})) = F(\vec{v}) + F(\vec{w})$

b. $F(k\vec{v}) = L(T(k\vec{v})) = L(kT(\vec{v})) = kL(T(\vec{v})) = kF(\vec{v})$

28. $T(\vec{e}_1) = B(A\vec{e}_1) = \begin{bmatrix} p & q \\ r & s \end{bmatrix}\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} pa + qc \\ ra + sc \end{bmatrix}$, and $T(\vec{e}_2) = B(A\vec{e}_2) = \begin{bmatrix} p & q \\ r & s \end{bmatrix}\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} pb + qd \\ rb + sd \end{bmatrix}$, so that the desired matrix is $\begin{bmatrix} pa + qc & pb + qd \\ ra + sc & rb + sd \end{bmatrix}$.

34. Keep in mind that the columns of the matrix of a linear transformation T are $T(\vec{e}_1)$, $T(\vec{e}_2)$, and $T(\vec{e}_3)$.

If T is the orthogonal projection onto a line L, then $T(\vec{x})$ will be on L for all $\vec{x}$ in $\mathbb{R}^3$; in particular, the three columns of the matrix of T will be on L, and therefore pairwise parallel. This is the case only for matrix B: B represents an orthogonal projection onto a line.
A reflection transforms orthogonal vectors into orthogonal vectors; therefore, the three columns of its matrix must be pairwise orthogonal. This is the case only for matrix E: E represents the reflection in a line.