

3.3

1. $\text{rref}(A) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

A basis of $\ker(A)$ is $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$; $\dim(\ker(A)) = 1$.

2. $\text{rref}(A) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

A basis of $\ker(A)$ is $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. Dimension is 2.

6. $\text{rref}(A) = I_3$

The basis of $\ker(A)$ is \emptyset , so that $\dim(\ker(A)) = 0$.

7. $\text{rref}(A) = A$

A basis of $\ker(A)$ is $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ -3 \\ 0 \\ -4 \\ 1 \end{bmatrix}$, so that $\dim(\ker(A)) = 3$.

8. $\text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 5 & -2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

A basis of $\ker(A)$ is $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -5 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$, so that $\dim(\ker(A)) = 3$.

9. $\text{rref}(A) = A$

A basis of $\ker(A)$ is $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, so that $\dim(\ker(A)) = 4$.

10. $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

A basis of $\ker(A)$ is $\begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$, so that $\dim(\ker(A)) = 1$.

11. A basis of $\text{im}(A)$ is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, so that $\dim(\text{im}(A)) = 1$ (see Exercise 1).

12. A basis of $\text{im}(A)$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, so that $\dim(\text{im}(A)) = 1$ (see Exercise 2).

13. A basis of $\text{im}(A)$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, so that $\dim(\text{im}(A)) = 2$ (see Exercise 4).

14. A basis of $\text{im}(A)$ is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$, so that $\dim(\text{im}(A)) = 2$ (see Exercise 4).

15. A basis of $\text{im}(A)$ is $[5]$ (or $[k]$ for any nonzero k), so that $\dim(\text{im}(A)) = 1$.

16. A basis of $\text{im}(A)$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$, so that $\dim(\text{im}(A)) = 3$ (see Exercise 6).

17. $\text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so that a basis of $\text{im}(A)$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, with $\dim(\text{im}(A)) = 2$.

18. $\text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, so that a basis of $\text{im}(A)$ is $\begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \\ 9 \\ 1 \end{bmatrix}$, with $\dim(\text{im}(A)) = 3$.

24. We form a 5×5 matrix A with the given vectors as its columns, and we find a basis of $\text{im}(A)$.

$$\text{rref}(A) = \begin{bmatrix} 1 & 3 & 0 & -1 & 3 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A basis of $\text{im}(A)$ is $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 4 \\ 1 \\ 2 \end{bmatrix}$.

26. Find a basis $\vec{v}_1, \vec{v}_2, \vec{v}_3$ of $\ker(A)$ and let $B = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$.

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \\ 0 & 0 & 1 \end{bmatrix}.$$

34. We can write $V = \ker(A)$, where A is the $m \times n$ matrix with entries a_{ij} . Note that $\text{rank}(A) \leq m$. Therefore, $\dim(V) = \dim(\ker(A)) = n - \text{rank}(A) \geq n - m$, by Fact 3.3.5.
35. We need to find all vectors \vec{x} in \mathbb{R}^n such that $\vec{v} \cdot \vec{x} = 0$, or $v_1x_1 + v_2x_2 + \cdots + v_nx_n = 0$, where the v_i are the components of the vector \vec{v} . These vectors form a hyperplane in \mathbb{R}^n (see Exercise 33), so that the dimension of the space is $n - 1$.
36. No; if $\text{im}(A) = \ker(A)$ for an $n \times n$ matrix A , then $n = \dim(\ker(A)) + \dim(\text{im}(A)) = 2 \dim(\text{im}(A))$, so that n is an even number.
37. Since $\dim(\ker(A)) = 5 - \text{rank}(A)$, any 4×5 matrix with rank 2 will do; for example,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$