3.4

1. We need to find the scalars c_1 and c_2 such that $\begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. Solving a linear system gives $c_1 = 3$, $c_2 = 4$. Thus $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

2. Proceeding as in Example 1, we find $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

3. Proceeding as in Example 1, we find $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$.

4. Proceeding as in Example 1, we find $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$.

12. Proceeding as in Exercise 11, we find the coordinate vector $\begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$.

13. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $S = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, $S^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$.

By Fact 3.4.4 the new matrix of T, namely B, is given by $B = S^{-1}AS = \begin{bmatrix} -1 & -1 \\ 4 & 6 \end{bmatrix}$.

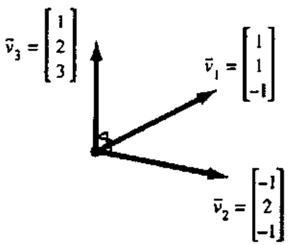
14. $A = \begin{bmatrix} 7 & -1 \\ -6 & 8 \end{bmatrix}$; $S = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$; $S^{-1} = \frac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$.

By Fact 3.4.4 the new matrix of T, namely B, is given by $B = S^{-1}AS = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}$.

15. a. Let B be the matrix of T with respect to the basis $\vec{v}_1 = \begin{bmatrix}1\\3\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}-3\\1\end{bmatrix}$. By Fact 3.4.3, $B = [[T(\vec{v}_1)]_{\mathcal{B}} \quad [T(\vec{v}_2)]_{\mathcal{B}}] = [[\vec{v}_1]_{\mathcal{B}} \quad [\vec{0}]_{\mathcal{B}}] = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$.

b. $A = SBS^{-1}$, with the matrix $B = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$ from part a, and $S = [\vec{v}_1 \; \vec{v}_2] = \begin{bmatrix}1 & -3\\3 & 1\end{bmatrix}$. A straightforward computation gives $A = \begin{bmatrix}0.1 & 0.3\\0.3 & 0.9\end{bmatrix}$.

16. a. $\begin{bmatrix}1\\1\\-1\end{bmatrix}$ and $\begin{bmatrix}-1\\2\\-1\end{bmatrix}$ are on the plane, and $\begin{bmatrix}1\\2\\3\end{bmatrix}$ is perpendicular to it.



Let B be the desired matrix. By Fact 3.4.3,

$$B = [[T(\vec{v}_1)]_{\mathcal{B}} \; [T(\vec{v}_2)]_{\mathcal{B}} \; [T(\vec{v}_3)]_{\mathcal{B}}] = [[\vec{v}_1]_{\mathcal{B}}[\vec{v}_2]_{\mathcal{B}}[\vec{0}]_{\mathcal{B}}] = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}.$$

b. By Fact 3.4.4, $A = SBS^{-1} = \begin{bmatrix}1 & -1 & 1\\1 & 2 & 2\\-1 & -1 & 3\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & -1 & 1\\1 & 2 & 2\\-1 & -1 & 3\end{bmatrix}^{-1} = \frac{1}{14}\begin{bmatrix}13 & -2 & -3\\-2 & 10 & -6\\-3 & -6 & 5\end{bmatrix}$.

17. By Fact 3.4.4, $B = S^{-1}AS = \begin{bmatrix}3 & 5\\5 & 8\end{bmatrix}^{-1}\begin{bmatrix}1 & 9\\9 & 4\end{bmatrix}\begin{bmatrix}3 & 5\\5 & 8\end{bmatrix} = \begin{bmatrix}-149 & -231\\99 & 154\end{bmatrix}$.

18. By Fact 3.4.4, $A = SBS^{-1} = \begin{bmatrix}3 & 5\\5 & 8\end{bmatrix}\begin{bmatrix}1 & 9\\9 & 7\end{bmatrix}\begin{bmatrix}3 & 5\\5 & 8\end{bmatrix}^{-1} = \begin{bmatrix}-74 & 54\\-111 & 82\end{bmatrix}$.

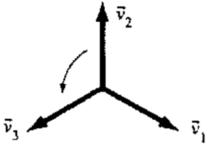
19. By Fact 3.4.4, $A = SBS^{-1} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}^{-1} = \begin{bmatrix}d & c\\b & a\end{bmatrix}$.

20. By Fact 3.4.1, $\begin{bmatrix}5\\3\end{bmatrix}_{\mathcal{B}} = \begin{bmatrix}3 & -1\\1 & 1\end{bmatrix}^{-1}\begin{bmatrix}5\\3\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$

36. We seek a basis $\vec{v}_1 = \begin{bmatrix} x \\ z \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} y \\ t \end{bmatrix}$ such that the matrix $S = [\vec{v}_1\ \vec{v}_2] = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ satisfies the equation $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$. Solving the ensuing linear system gives $S = \begin{bmatrix} \frac{z}{2} & -t \\ z & t \end{bmatrix}$. We need to choose both z and t nonzero to make S invertible. For example, if we let $z = 2$ and $t = 1$, then $S = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$, so that $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

47. a. By inspection, we can find an orthonormal basis $\vec{v}_1 = \vec{v}, \vec{v}_2, \vec{v}_3$ of $\mathbb{R}^3$:

$$\vec{v}_1 = \vec{v} = \begin{bmatrix} 0.6 \\ 0.8 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0.8 \\ -0.6 \\ 0 \end{bmatrix}$$



b. Now $T(\vec{v}_1) = \vec{v}_1, T(\vec{v}_2) = \vec{v}_3$ and $T(\vec{v}_3) = -\vec{v}_2$, so that the matrix B of T with respect to the basis $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}. \text{ Then } A = SBS^{-1} = \begin{bmatrix} 0.36 & 0.48 & 0.8 \\ 0.48 & 0.64 & -0.6 \\ -0.8 & 0.6 & 0 \end{bmatrix}.$$

48. a. $\vec{v}_0 + \vec{v}_1 + \vec{v}_2 + \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

b. If $\mathcal{B}$ is the basis $\vec{v}_1, \vec{v}_2, \vec{v}_3$, then $\vec{v}_0 + \vec{v}_1 + \vec{v}_2 + \vec{v}_3 = \vec{0}$ (by part a) so $\vec{v}_0 = -\vec{v}_1 - \vec{v}_2 - \vec{v}_3$, i.e. $[\vec{v}_0]_{\mathcal{B}} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$.

c. $T(\vec{v}_2) = T(-\vec{v}_0 - \vec{v}_1 - \vec{v}_3) = -T(\vec{v}_0) - T(\vec{v}_1) - T(\vec{v}_3) = -\vec{v}_3 - \vec{v}_0 - \vec{v}_1 = \vec{v}_2$

Hence, T is a rotation through 120° about the line spanned by $\vec{v}_2$. Its matrix, B, is given by $[[T(\vec{v}_1)]_{\mathcal{B}}[T(\vec{v}_2)]_{\mathcal{B}}[T(\vec{v}_3)]_{\mathcal{B}}]$ where

$$T(\vec{v}_1) = \vec{v}_0 = -\vec{v}_1 - \vec{v}_2 - \vec{v}_3 \text{ so } [T(\vec{v}_1)]_{\mathcal{B}} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

$$T(\vec{v}_2) = \vec{v}_2 \text{ so } [T(\vec{v}_2)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$T(\vec{v}_3) = \vec{v}_1 \text{ so } [T(\vec{v}_3)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and $B = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

$B^3 = I_3$ since if the tetrahedron rotates through 120° three times, it returns to the original position.