4.2

1. Fails to be linear, since $T(A+B) = A + B + I_2$ doesn't equal $T(A) + T(B) = A + I_2 + B + I_2 = A + B + 2I_2$.

2. Linear, since $T(A+B) = 7(A+B) = 7A + 7B$ equals $T(A) + T(B) = 7A + 7B$, and $T(kA) = 7kA$ equals $kT(A) = k(7A) = 7kA$.
   Yes, T is an isomorphism, with $T^{-1}(A) = \frac{1}{7}A$.

3. Linear, since $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} p & q \\ r & s \end{bmatrix}\right) = T\begin{bmatrix} a+p & b+q \\ c+r & d+s \end{bmatrix} = a + p + d + s$

equals $T\begin{bmatrix} a & b \\ c & d \end{bmatrix} + T\begin{bmatrix} p & q \\ r & s \end{bmatrix} = a+d+p+s$, and $T\left(k\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = T\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix} = ka+kd$

equals $kT\begin{bmatrix} a & b \\ c & d \end{bmatrix} = k(a+d) = ka+kd$.

No, T fails to be an isomorphism, since $4 = \dim(\mathbb{R}^{2\times 2}) \neq \dim(\mathbb{R}) = 1$; see Fact 4.2.3d.

4. Fails to be linear, since $T(2I_2) = \det(2I_2) = 4$ does not equal $2T(I_2) = 2\det(I_2) = 2$.

14. Linear, since $T(f(t)+g(t)) = f(-t)+g(-t)$ equals
$T(f(t))+T(g(t)) = f(-t)+g(-t)$, and $T(kf(t)) = kf(-t)$ equals $kT(f(t)) = kf(-t)$.
Yes, T is an isomorphism; it's its own inverse, since $T(T(f(t))) = T(f(-t)) = f(t)$.

15. Linear, since $T(f(t)+g(t)) = f(2t)+g(2t)$ equals
$T(f(t))+T(g(t)) = f(2t)+g(2t)$, and $T(kf(t)) = kf(2t)$ equals $kT(f(t)) = kf(2t)$.
Yes, T is an isomorphism; the inverse is $T^{-1}(f(t)) = f\left(\frac{t}{2}\right)$.

16. Linear, since $T(f(t)+g(t)) = t(f'(t)+g'(t)) = t(f'(t))+t(g'(t))$ equals
$T(f(t))+T(g(t)) = t(f'(t))+t(g'(t))$, and $T(kf(t)) = t(kf'(t)) = kt(f'(t))$
equals $kT(f(t)) = kt(f'(t))$.
No, T isn't an isomorphism, since the constant function $f(t) = 1$ is in $\ker(T)$.

23. Linear, as in Exercise 16. Not an isomorphism, since the constant function $f(t) = 1$ isn't in the image.

24. Linear. Not an isomorphism, since the constant function $f(t) = 1$ is in the kernel.

35. The kernel consists of all polynomials $f(t)$ such that $t(f(t)) = 0$, that is, the zero polynomial $f(t) = 0$ alone.
The image consists of all polynomials $g(t)$ that can be written as $g(t) = t(f(t))$, meaning that we can factor out a t. These are the polynomials with constant term 0, of the form $g(t) = a_1t + a_2t^2 + \cdots + a_nt^n$.

36. The kernel consists of all polynomials with derivative 0, that is, all constant polynomials, of the form $f(t) = a$.
The image is all of P, since every polynomial $g(t)$ has an antiderivative $G(t)$ that is itself a polynomial, so that $g(t) = T(G(t))$.

40. The kernel of T consists of all smooth functions $f(t)$ such that $T(f(t)) = f(t) - f'(t) = 0$, or $f'(t) = f(t)$. As you may recall from a discussion of exponential functions in calculus, those are the functions of the form $f(t) = Ce^t$, where C is a constant. Thus the nullity of T is 1.

41. To see that $0 + 0 = 0$, apply the formula $f + 0 = f$ to $f = 0$ (see Definition 4.1.1). Then $0 = k0 - k0 = k(0 + 0) - k0 = k0 + k0 - k0 = k0$.