4.3

1. Let $\mathcal{B}$ be the standard basis of $P_2 : 1, t, t^2$. Then the coordinates of the given polynomials with respect to $\mathcal{B}$ are

$[f]_{\mathcal{B}} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}, [g]_{\mathcal{B}} = \begin{bmatrix} 9 \\ 9 \\ 4 \end{bmatrix}, [h]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$. Finding $\text{rref}\begin{bmatrix} 7 & 9 & 3 \\ 3 & 9 & 2 \\ 1 & 4 & 1 \end{bmatrix} = I_3$, we conclude that $[f]_{\mathcal{B}}, [g]_{\mathcal{B}}, [h]_{\mathcal{B}}$ are linearly independent, hence so are f, g, h, since the coordinate transformation is an isomorphism.

2. Let $\mathcal{B}$ be the basis $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ of $\mathbb{R}^{2\times 2}$. Then the coordinates of the given matrices with respect to $\mathcal{B}$ are $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ 6 & 8 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 1 \\ 4 \\ 6 \\ 8 \end{bmatrix}$. Finding $\text{rref}\begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 6 \\ 1 & 4 & 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \neq I_4$, we conclude that the four vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 6 \\ 8 \end{bmatrix}$ are linearly dependent, and so are the four given matrices. In fact $\begin{bmatrix} 1 & 4 \\ 6 & 8 \end{bmatrix} = -\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + 4\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$.

3. We proceed as in Exercise 1. Since $\text{rref}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 7 & 8 & 8 \\ 9 & 0 & 1 & 4 \\ 1 & 7 & 5 & 8 \end{bmatrix} = I_4$, the four given polynomials do form a basis of P_3.

4. Consider the coordinate vectors of the 3 given polynomials with respect to the standard basis of P_2: $1, t, t^2$.

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2k \\ 2+k \\ 1 \end{bmatrix}$$

Since matrix $\begin{bmatrix} 1 & 0 & 2k \\ 1 & 1 & 2+k \\ 0 & 1 & 1 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & 2k \\ 0 & 1 & 2-k \\ 0 & 0 & k-1 \end{bmatrix}$, these three vectors form a basis of $\mathbb{R}^3$ unless $k = 1$. Therefore, the three polynomials $f(t), tf(t), g(t)$ form a basis of P_2 unless $k = 1$.

5. Use a diagram as in Definition 4.3.2:

$$\begin{array}{ccc}
\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} & \xrightarrow{T} & \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} a & b+2c \\ 0 & 3c \end{bmatrix} \\
\downarrow & & \downarrow \\
\begin{bmatrix} a \\ b \\ c \end{bmatrix} & \xrightarrow{B} & \begin{bmatrix} a \\ b+2c \\ 3c \end{bmatrix}
\end{array}$$

The matrix B that transforms $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ into $\begin{bmatrix} a \\ b+2c \\ 3c \end{bmatrix}$ is $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$.

13. Use a diagram as in Definition 4.3.2:

$$\begin{array}{ccc}
\begin{bmatrix} a & b \\ c & d \end{bmatrix} & \xrightarrow{T} & \begin{bmatrix} a+c & b+d \\ 2a+2c & 2b+2d \end{bmatrix} \\
\downarrow & & \downarrow \\
\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} & \xrightarrow{B} & \begin{bmatrix} a+c \\ b+d \\ 2a+2c \\ 2b+2d \end{bmatrix}
\end{array}$$

The matrix is $B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}$.

14. Use Fact 4.3.3 to construct matrix B column by column:

$$B = \left[\left[\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}\right]_{\mathcal{B}} \left[\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}\right]_{\mathcal{B}} \left[\begin{matrix} 3 & 0 \\ 6 & 0 \end{matrix}\right]_{\mathcal{B}} \left[\begin{matrix} 0 & 3 \\ 0 & 6 \end{matrix}\right]_{\mathcal{B}} \right] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

21.

$$\begin{array}{ccc} a + bt + ct^2 & \xrightarrow{T} & b - 3a + (2c - 3b)t - 3ct^2 \\ \downarrow & & \downarrow \\ \begin{bmatrix} a \\ b \\ c \end{bmatrix} & \xrightarrow{B} & \begin{bmatrix} b - 3a \\ 2c - 3b \\ -3c \end{bmatrix} \end{array}$$

Thus $B = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 2 \\ 0 & 0 & -3 \end{bmatrix}$.

22.

$$\begin{array}{ccc} a + bt + ct^2 & \xrightarrow{T} & 4b + 2c + 8ct \\ \downarrow & & \downarrow \\ \begin{bmatrix} a \\ b \\ c \end{bmatrix} & \xrightarrow{B} & \begin{bmatrix} 4b + 2c \\ 8c \\ 0 \end{bmatrix} \end{array}$$

Thus $B = \begin{bmatrix} 0 & 4 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix}$.

38. In Exercise 22, we found $B = \begin{bmatrix} 0 & 4 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix}$, with rref $B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Now

$\text{im}(B) = \text{span}(\vec{e}_1, \vec{e}_2)$, and $\ker(B) = \text{span}(\vec{e}_1)$. Thus $\text{im}(T) = \text{span}(1, t)$, $\text{rank}(T) = 2$, and $\ker(T) = \text{span}(1)$. Note that $\text{im}(T)$ consists of the linear functions, and $\ker(T)$ consists of the constant functions.

50. $B = [[(b-1)\cos(t) - a\sin(t)]_{\mathcal{B}} \ \ [a\cos(t) + (b-1)\sin(t)]_{\mathcal{B}}] = \begin{bmatrix} b-1 & a \\ -a & b-1 \end{bmatrix}$.

51. Note that $\cos(t-\pi/2) = \sin(t)$ and $\sin(t-\pi/2) = -\cos(t)$. Thus

$$B = [[\cos(t-\pi/2)]_{\mathcal{B}} \ \ [\sin(t-\pi/2)]_{\mathcal{B}}] = [[\sin(t)]_{\mathcal{B}} \ \ [-\cos(t)]_{\mathcal{B}}] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

52. Recall that $\cos(t-\delta) = \cos(\delta)\cos(t) + \sin(\delta)\sin(t)$ and
$\sin(t-\delta) = \cos(\delta)\sin(t) - \sin(\delta)\cos(t)$. Also, $\cos(\pi/4) = \sin(\pi/4) = \sqrt{2}/2$.
Thus
$B = [[\cos(t-\pi/4)]_{\mathcal{B}} \ \ [\sin(t-\pi/4)]_{\mathcal{B}}]$
$= \left[\left[\frac{\sqrt{2}}{2}\cos(t) + \frac{\sqrt{2}}{2}\sin(t)\right]_{\mathcal{B}} \ \ \left[-\frac{\sqrt{2}}{2}\cos(t) + \frac{\sqrt{2}}{2}\sin(t)\right]_{\mathcal{B}}\right] = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

53. Recall that $\cos(t-\delta) = \cos(\delta)\cos(t) + \sin(\delta)\sin(t)$ and
$\sin(t-\delta) = \cos(\delta)\sin(t) - \sin(\delta)\cos(t)$.
Thus

$B = [[\cos(t-\delta)]_{\mathcal{B}} \ \ [\sin(t-\delta)]_{\mathcal{B}}]$

$= [[\cos(\delta)\cos(t) + \sin(\delta)\sin(t)]_{\mathcal{B}} \ \ [-\sin(\delta)\cos(t) + \cos(\delta)\sin(t)]_{\mathcal{B}}] = \begin{bmatrix} \cos(\delta) & -\sin(\delta) \\ \sin(\delta) & \cos(\delta) \end{bmatrix}$.

Note that B is a rotation matrix.

54. Note that the two basis vectors $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$ are perpendicular. Thus $T\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and

$T\begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Now $B = \left[\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}_{\mathcal{B}} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_{\mathcal{B}}\right] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.