5. Yes, since the product of orthogonal matrices is orthogonal, by Fact 5.3.4a.

6. Yes! By Facts 5.3.9b and 5.3.7, $(A^T)^{-1} = (A^{-1})^T = (A^T)^T$.
The equation $(A^T)^{-1} = (A^T)^T$ shows that A^T is orthogonal, again by Fact 5.3.7.

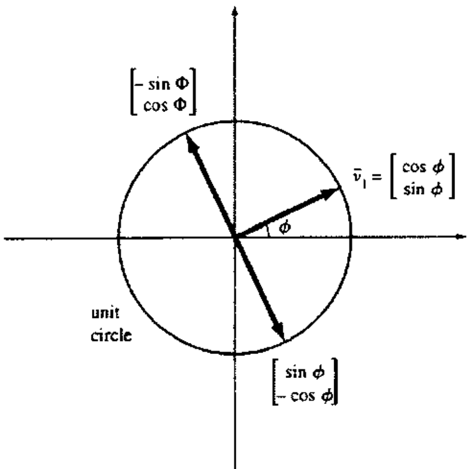
7. Yes! If A is orthogonal, then so is A^T, by Exercise 6. Since the columns of A are orthonormal, so are the rows of A^T.

8. a. No! As a counterexample, consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ (see Exercise 4).

b. Yes! More generally, if A and B are $n \times n$ matrices such that $BA = I_n$, then $AB = I_n$, by Fact 2.4.9c.

9. Write $A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}$. The unit vector $\vec{v}_1$ can be expressed as $\vec{v} = \begin{bmatrix} \cos(\phi) \\ \sin(\phi) \end{bmatrix}$, for some ϕ. Then $\vec{v}_2$ will be one of the two unit vectors orthogonal to $\vec{v}_1$: $\vec{v}_2 = \begin{bmatrix} -\sin(\phi) \\ \cos(\phi) \end{bmatrix}$ or $\vec{v}_2 = \begin{bmatrix} \sin(\phi) \\ -\cos(\phi) \end{bmatrix}$.

Therefore, an orthogonal 2×2 matrix is either of the form $A = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$ or $A = \begin{bmatrix} \cos(\phi) & \sin(\phi) \\ \sin(\phi) & -\cos(\phi) \end{bmatrix}$, representing a rotation or a reflection.



10. Since the first two columns are orthogonal to the third, we have $c = d = 0$. Then $\begin{bmatrix} a & b \\ e & f \end{bmatrix}$ is an orthogonal 2×2 matrix; By Exercise 9, the 3×3 matrix A is either of the form $A = \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ 0 & 0 & 1 \\ \sin(\phi) & \cos(\phi) & 0 \end{bmatrix}$ or $A = \begin{bmatrix} \cos(\phi) & \sin(\phi) & 0 \\ 0 & 0 & 1 \\ \sin(\phi) & -\cos(\phi) & 0 \end{bmatrix}$.

11. Let us first think about the inverse $L = T^{-1}$ of T.

Write $L(\vec{x}) = A\vec{x} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} \vec{x}$. It is required that $L(\vec{e}_3) = \vec{v}_3 = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$.

Furthermore, the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ must form an orthonormal basis of $\mathbb{R}^3$. By inspection, we find $\vec{v}_1 = \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$.

Then $\vec{v}_2 = \vec{v}_1 \times \vec{v}_3 = \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix}$ does the job. In summary, we have $L(\vec{x}) = \frac{1}{3}\begin{bmatrix} -2 & -1 & 2 \\ 1 & 2 & 2 \\ 2 & -2 & 1 \end{bmatrix} \vec{x}$.

Since the matrix of L is orthogonal, the matrix of $T = L^{-1}$ is the transpose of the matrix of L:

$T(\vec{x}) = \frac{1}{3}\begin{bmatrix} -2 & 1 & 2 \\ -1 & 2 & -2 \\ 2 & 2 & 1 \end{bmatrix} \vec{x}$.

There are many other answers (since there are many choices for the vector $\vec{v}_1$ above).

12. Let the third column be the cross product of the first two: $A = \begin{bmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} \\ \frac{2}{3} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} \\ \frac{1}{3} & 0 & -\frac{4}{\sqrt{18}} \end{bmatrix}$.

There is another solution, with the signs in the last column reversed.

13. No, since the vectors $\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}$ are orthogonal, whereas $\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$ are not (see Fact 5.3.2).

14. By Fact 5.3.9a, $(A^T A)^T = A^T (A^T)^T = A^T A$.
The matrix AA^T is symmetric as well: $(AA^T)^T = (A^T)^T A^T = AA^T$.

15. No! If two symmetric matrices A and B do not commute, then $(AB)^T = B^T A^T = BA \neq AB$, so that AB is not symmetric.
Example: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

16. Yes, since $(A^2)^T = (AA)^T = A^T A^T = AA = A^2$.

17. Yes! By Fact 5.3.9b, $(A^{-1})^T = (A^T)^{-1} = A^{-1}$.

20. An orthonormal basis of W is $\vec{w}_1 = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}, \vec{w}_2 = \begin{bmatrix} -0.1 \\ 0.7 \\ -0.7 \\ 0.1 \end{bmatrix}$ (see Exercise 5.2.9).

By Fact 5.3.10, the matrix of the projection onto W is AA^T, where $A = \begin{bmatrix} \vec{w}_1 & \vec{w}_2 \end{bmatrix}$.

$$AA^T = \frac{1}{100} \begin{bmatrix} 26 & 18 & 32 & 24 \\ 18 & 74 & -24 & 32 \\ 32 & -24 & 74 & 18 \\ 24 & 32 & 18 & 26 \end{bmatrix}$$