

3. Note that  $\langle \vec{x}, \vec{y} \rangle = (S\vec{x})^T S\vec{y} = S\vec{x} \cdot S\vec{y}$ .

a. We will check the four parts of Definition 5.5.1

$\alpha.$   $\langle \vec{x}, \vec{y} \rangle = S\vec{x} \cdot S\vec{y} = S\vec{y} \cdot S\vec{x} = \langle \vec{y}, \vec{x} \rangle$

$\beta.$   $\langle \vec{x} + \vec{y}, \vec{z} \rangle = S(\vec{x} + \vec{y}) \cdot S\vec{z} = (S\vec{x} + S\vec{y}) \cdot S\vec{z} = (S\vec{x} \cdot S\vec{z}) + (S\vec{y} \cdot S\vec{z}) = \langle \vec{x}, \vec{z} \rangle + \langle \vec{y}, \vec{z} \rangle$

$\gamma.$   $\langle c\vec{x}, \vec{y} \rangle = S(c\vec{x}) \cdot S\vec{y} = c(S\vec{x}) \cdot S\vec{y} = c\langle \vec{x}, \vec{y} \rangle$

$\delta.$  If  $\vec{x} \neq \vec{0}$ , then  $\langle \vec{x}, \vec{x} \rangle = S\vec{x} \cdot S\vec{x} = \|S\vec{x}\|^2$  is positive if  $S\vec{x} \neq \vec{0}$ , that is, if  $\vec{x}$  is not in the kernel of  $S$ . It is required that  $S\vec{x} \neq \vec{0}$  whenever  $\vec{x} \neq \vec{0}$ , that is,  $\ker(S) = \{\vec{0}\}$ .

*Answer:*  $S$  must be invertible.

b. It is required that  $\langle \vec{x}, \vec{y} \rangle = (S\vec{x})^T S\vec{y} = \vec{x}^T S^T S\vec{y}$  equal  $\vec{x} \cdot \vec{y} = \vec{x}^T \vec{y}$  for all  $\vec{x}$  and  $\vec{y}$ . This is the case if and only if  $S^T S = I_n$ , that is,  $S$  is orthogonal.

6. a. The  $ii$ th entry of  $PQ$  is  $\sum_{k=1}^n p_{ik}q_{ki}$ , so that  $\text{tr}(PQ) = \sum_{i=1}^m \sum_{k=1}^n p_{ik}q_{ki}$ .

Likewise,  $\text{tr}(QP) = \sum_{i=1}^n \sum_{k=1}^m q_{ik}p_{ki}$ .

Reversing the roles of  $i$  and  $k$  and the order of summation we see that  $\text{tr}(PQ) = \text{tr}(QP)$ , as claimed.

b. Using part a and the fact that  $\text{tr}(M) = \text{tr}(M^T)$  for any square matrix  $M$ , we find that  $\langle A, B \rangle = \text{tr}(A^T B) = \text{tr}(BA^T) = \text{tr}((BA^T)^T) = \text{tr}(AB^T) = \langle\langle A, B \rangle\rangle$

7. Axioms  $a$ ,  $b$ , and  $c$  hold for any choice of  $k$  (check this!). Also, it is required that  $\langle\langle v, v \rangle\rangle = k\langle v, v \rangle$  be positive for nonzero  $v$ . Since  $\langle v, v \rangle$  is positive, this is the case if (and only if)  $k$  is *positive*.

8. By parts b and c of Definition 9.3.1, we have  $T(u+v) = \langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle = T(u) + T(v)$  and  $T(cv) = \langle cv, w \rangle = c\langle v, w \rangle = cT(v)$ , so that  $T$  is linear. If  $w = 0$ , then  $\text{im}(T) = \{0\}$  and  $\text{ker}(T) = V$ . If  $w \neq 0$ , then  $\text{im}(T) = \mathbb{R}$  and  $\text{ker}(T)$  consists of all  $v$  perpendicular to  $w$ .

9. If  $f$  is even and  $g$  is odd, then  $fg$  is odd, so that  $\langle f, g \rangle = \int_{-1}^1 fg = 0$ .

10. A function  $g(t) = a + bt + ct^2$  is orthogonal to  $f(t) = t$  if

$$\langle f, g \rangle = \int_{-1}^1 (at + bt^2 + ct^3) dt = \left[ \frac{a}{2}t^2 + \frac{b}{3}t^3 + \frac{c}{4}t^4 \right]_{-1}^1 = \frac{2}{3}b = 0, \text{ that is, if } b = 0.$$

Thus, the functions 1 and  $t^2$  form a basis of the space of all functions in  $P_2$  orthogonal to  $f(t) = t$ . To find an *orthonormal* basis  $g_1(t), g_2(t)$ , we apply Gram-Schmidt. Now  $\|1\| = \frac{1}{2} \int_{-1}^1 1 dt = 1$ , so that we

can let  $g_1(t) = 1$ . Then  $g_2(t) = \frac{t^2 - \langle 1, t^2 \rangle 1}{\|t^2 - \langle 1, t^2 \rangle 1\|} = \frac{t^2 - \frac{1}{3}}{\|t^2 - \frac{1}{3}\|} = \frac{\sqrt{5}}{2}(3t^2 - 1)$

*Answer:*  $g_1(t) = 1, g_2(t) = \frac{\sqrt{5}}{2}(3t^2 - 1)$

11.  $\langle f, g \rangle = \langle \cos(t), \cos(t + \delta) \rangle = \langle \cos(t), \cos(t)\cos(\delta) - \sin(t)\sin(\delta) \rangle = \cos(\delta)\langle \cos(t), \cos(t) \rangle - \sin(\delta)\langle \cos(t), \sin(t) \rangle = \cos(\delta)$ , by Fact 5.5.4.

Also,  $\langle g, g \rangle = 1$  (left to reader).

Thus,  $\angle(f, g) = \arccos\left(\frac{\langle f, g \rangle}{\|f\|\|g\|}\right) = \arccos(\cos \delta) = \delta$ .

12. By Fact 5.5.5,  $a_0 = \left\langle |t|, \frac{1}{\sqrt{2}} \right\rangle = \frac{1}{\sqrt{2}\pi} \int_{-\pi}^{\pi} |t| dt = \frac{\pi}{\sqrt{2}}$ ,  $b_k = \langle |t|, \sin(kt) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} |t| \sin(kt) dt = 0$ , since the integrand is an odd function.

$$c_k = \langle |t|, \cos(kt) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} |t| \cos(kt) dt = \frac{2}{\pi} \int_0^{\pi} t \cos(kt) dt = \frac{2}{\pi} \left\{ \left[ \frac{1}{k} t \sin(kt) \right]_0^{\pi} - \int_0^{\pi} \frac{1}{k} \sin(kt) dt \right\}$$

$$= \frac{2}{\pi k^2} [\cos(kt)]_0^{\pi} = \frac{2}{\pi k^2} (\cos(k\pi) - 1) = \begin{cases} 0 & \text{if } k \text{ is even} \\ -\frac{4}{k^2\pi} & \text{if } k \text{ is odd} \end{cases}$$

*Summary:*

$$a_0 = 0$$

$$b_k = 0$$

$$c_k = \begin{cases} 0 & \text{if } k \text{ is even} \\ -\frac{4}{k^2 \pi} & \text{if } k \text{ is odd} \end{cases}$$

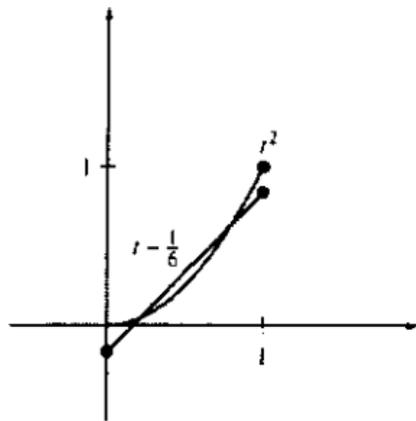
16. a. We start with the standard basis  $1, t$  and use the Gram-Schmidt process to construct an *orthonormal* basis  $g_1(t), g_2(t)$ .

$$\|1\| = \sqrt{\int_0^1 dt} = 1, \text{ so that we can let } g_1(t) = 1. \text{ Then } g_2(t) = \frac{t - \langle 1, t \rangle 1}{\|t - \langle 1, t \rangle 1\|} = \frac{t - \frac{1}{2}}{\|t - \frac{1}{2}\|} = \sqrt{3}(2t - 1).$$

*Summary:*  $g_1(t) = 1$  and  $g_2(t) = \sqrt{3}(2t - 1)$  is an orthonormal basis.

- b. We are looking for  $\text{proj}_{P_1}(t^2) = \langle g_1(t), t^2 \rangle g_1(t) + \langle g_2(t), t^2 \rangle g_2(t)$ , by Fact 5.5.3.

We find that  $\langle g_1(t), t^2 \rangle = \int_0^1 t^2 dt = \frac{1}{3}$  and  $\langle g_2(t), t^2 \rangle = \sqrt{3} \int_0^1 (2t^3 - t^2) dt = \frac{\sqrt{3}}{6}$ , so that  $\text{proj}_{P_1} t^2 = \frac{1}{3} + \frac{1}{2}(2t - 1) = t - \frac{1}{6}$ .



19. If we write  $\vec{v} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ ,  $\vec{w} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ , and  $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ , then  $\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle = [x_1 \ x_2] \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = px_1y_1 + qx_1y_2 + rx_2y_1 + sx_2y_2$ . Note that in Exercise 15 we considered the special case  $p = 1$ . First it is required that  $p = \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle$  be positive.

Now we can write  $\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle = p \left[ x_1y_1 + \frac{q}{p}x_1y_2 + \frac{r}{p}x_2y_1 + \frac{s}{p}x_2y_2 \right]$  and use our work in Exercise 15 (with  $b = \frac{q}{p}$ ,  $c = \frac{r}{p}$ ,  $d = \frac{s}{p}$ ) to see that the conditions  $q = r$  and  $q^2 < ps$  must hold. In summary, the function is an inner product if (and only if) the entries of matrix  $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$  satisfy the conditions  $p > 0$ ,  $q = r$  and  $\det(A) = ps - q^2 > 0$ .

20. a.  $\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle = [x_1 \ x_2] \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = x_1 + 2x_2 = 0$  when  $x_1 = -2x_2$ . This is the line spanned by vector  $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ .

b. Since vectors  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$  are orthogonal, we merely have to multiply each of them with the reciprocal of its norm. Now  $\left\| \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\|^2 = [1 \ 0] \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1$ , so that  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  is a unit vector, and  $\left\| \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\|^2 = [-2 \ 1] \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = 4$ , so that  $\left\| \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\| = 2$ . Thus  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix}$  is an orthonormal basis.