1. $\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix} = -3$

2. $\det \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 9 & 1 & 0 \\ 0 & 9 & 0 & 0 \\ 1 & 9 & 9 & 5 \end{bmatrix} = -45$

3. $\det \begin{bmatrix} 1 & -1 & 2 & -2 \\ -1 & 2 & 1 & 6 \\ 2 & 1 & 14 & 10 \\ -2 & 6 & 10 & 33 \end{bmatrix} = 9$

6. There are many ways to do this problem; here is one possible approach:
Subtracting the second to last row from the last, we can make the last row into $[0 \quad 0 \quad \cdots \quad 0 \quad 1]$.
Now expanding along the last row we see that $\det(M_n) = \det(M_{n-1})$.
Since $\det(M_1) = 1$ we can conclude that $\det(M_n) = 1$ for all n.

7. $\det(A) = 1$

8. Since $\vec{v}_2, \ldots, \vec{v}_n$ are linearly independent, $T(\vec{x}) = 0$ only if $\vec{x}$ is a linear combination of the $\vec{v}_i$'s, (otherwise the matrix $[\vec{x}\, \vec{v}_2 \cdots \vec{v}_n]$ is invertible, and $T(\vec{x}) \neq 0$). Hence, the kernel of T is the span of $\vec{v}_2, \ldots, \vec{v}_n$, an $(n-1)$-dimensional subspace of $\mathbb{R}^n$. The image of T is the real line $\mathbb{R}$ (since it must be 1-dimensional).

26. By Exercise 25, $\det(A^TA) = [\det(A)]^2$. Since A is orthogonal, $A^TA = I_n$ so that $1 = \det(I_n) = \det\,(A^TA) = [\det(A)]^2$ and $\det(A) = \pm 1$.

27. $\det(A) = \det(A^T) = \det(-A) = (-1)^n(\det A) = -\det(A)$, so that $\det(A) = 0$. We have used Facts 6.2.1 and 6.2.3.b.

28. $$\det(A^TA) \underset{\substack{\uparrow \\ \text{Definition of } A}}{=} \det((QR)^TQR) = \det(R^TQ^TQR) \underset{\substack{\uparrow \\ \text{Since columns of } Q \text{ orthonormal}}}{=} \det(R^TI_nR) = \det(R^TR) \underset{\substack{\uparrow \\ \text{Fact 6.2.7}}}{=} \det(R^T)\det(R)$$

$$\underset{\substack{\uparrow \\ \text{Fact} \\ 6.2.1}}{=} [\det(R)]^2 \underset{\substack{\uparrow \\ \text{Since } R \\ \text{is triangular.}}}{=} \left(\prod_{i=1}^{n} r_{ii}\right)^2 > 0$$

29. $\det(A^TA) = \det\left(\begin{bmatrix} \vec{v}^T \\ \vec{w}^T \end{bmatrix}\begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix}\right) = \det\begin{bmatrix} \vec{v}\cdot\vec{v} & \vec{v}\cdot\vec{w} \\ \vec{v}\cdot\vec{w} & \vec{w}\cdot\vec{w} \end{bmatrix} = \det\begin{bmatrix} \|\vec{v}\|^2 & \vec{v}\cdot\vec{w} \\ \vec{v}\cdot\vec{w} & \|\vec{w}\|^2 \end{bmatrix} = \|\vec{v}\|^2\|\vec{w}\|^2 - (\vec{v}\cdot\vec{w})^2 \geq 0$ by the Cauchy-Schwarz inequality (Fact 5.1.10).

30. a. We claim that $\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n \neq \vec{0}$ if and only if the vectors $\vec{v}_2, \ldots, \vec{v}_n$ are linearly independent. If the vectors $\vec{v}_2, \ldots, \vec{v}_n$ are linearly independent, then we can find a basis $\vec{x}, \vec{v}_2, \ldots, \vec{v}_n$ of $\mathbb{R}^n$ (any vector $\vec{x}$ that is not in span $(\vec{v}_2, \ldots, \vec{v}_n)$ will do). Then $\vec{x}\cdot(\vec{v}_2 \times \cdots \times \vec{v}_n) = \det[\vec{x}\,\vec{v}_2 \cdots \vec{v}_n] \neq 0$, so that $\vec{v}_2 \times \cdots \times \vec{v}_n \neq \vec{0}$. Conversely, suppose that $\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n \neq 0$; say the ith component of this vector is nonzero. Then $0 \neq \vec{e}_i \cdot (\vec{v}_2 \times \cdots \times \vec{v}_n) = \det[\vec{e}_i\,\vec{v}_2 \cdots \vec{v}_n]$, so that the vectors $\vec{v}_2, \ldots, \vec{v}_n$ are linearly independent (being columns of an invertible matrix).

b. ith component of $\vec{e}_2 \times \vec{e}_3 \times \cdots \times \vec{e}_n = \det \begin{bmatrix} 1 & 1 & & 1 \\ \vec{e}_i & \vec{e}_2 & \cdots & \vec{e}_n \\ 1 & 1 & & 1 \end{bmatrix} = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{if } i > 1 \end{cases}$

so $\vec{e}_2 \times \vec{e}_3 \times \cdots \times \vec{e}_n = \vec{e}_1$.

c. $\vec{v}_i \cdot (\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n) = \det \begin{bmatrix} 1 & 1 & 1 & & 1 \\ \vec{v}_i & \vec{v}_2 & \vec{v}_3 & \cdots & \vec{v}_n \\ 1 & 1 & 1 & & 1 \end{bmatrix} = 0$

for any $2 \le i \le n$ since the above matrix has two identical columns.

d. Compare the ith components of the two vectors:

$$\det \begin{bmatrix} 1 & 1 & 1 & & 1 \\ \vec{e}_i & \vec{v}_2 & \vec{v}_3 & \cdots & \vec{v}_n \\ 1 & 1 & 1 & & 1 \end{bmatrix} \text{ and } \det \begin{bmatrix} 1 & 1 & 1 & & 1 \\ \vec{e}_i & \vec{v}_3 & \vec{v}_2 & \cdots & \vec{v}_n \\ 1 & 1 & 1 & & 1 \end{bmatrix}.$$

The two determinants differ by a factor of -1 by Fact 6.2.4b, so that $\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n = -\vec{v}_3 \times \vec{v}_2 \times \cdots \times \vec{v}_n$.

e. $\det[\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n \ \vec{v}_2 \ \vec{v}_3 \cdots \vec{v}_n] = (\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n) \cdot (\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n) = \|\vec{v}_2 \times \cdots \times \vec{v}_n\|^2$

f. On page 243 of the text we saw that the "old" cross product satisfies the defining equation of the "new" cross product: $\vec{x} \cdot (\vec{v}_2 \times \vec{v}_3) = \det \left[\vec{x} \ \vec{v}_2 \ \vec{v}_3 \right]$.

34. a. $D_1 = 2, D_2 = 3, D_3 = 4, D_4 = 5$

b. $D_n = 2D_{n-1} - D_{n-2}$ (expand along the first column to see this)

c. $D_n = n + 1$ (by induction)

35. $\det(Q_1) = \det(Q_2) = \det(Q_3) = 1$
$\det(Q_n) = 2\det(Q_{n-1}) - \det(Q_{n-2})$ (expand along the first column), so that $\det(Q_n) = 1$ for all n.

41. Mimicking the proof of Fact 6.2.4c we can show:
If B is obtained from A by adding a multiple of a row of A to another row, then $D(B) = D(A)$. As in the proof of Algorithm 6.2.6 it follows that $D(A) = (-1)^s k_1 k_2 \ldots k_r D(\text{rref } A)$.
If A is not invertible, then $D(\text{rref } A) = 0$, by the linearity of D in the last row, so that $D(A) = \det(A) = 0$. If A is invertible, then $\text{rref}(A) = I_n$ and $D(\text{rref } A) = 1$, by property c of the function D. Therefore, $D(A) = (-1)^s k_1 k_2 \cdots k_r = \det(A)$.
We have shown that $D(A) = \det(A)$ for all $n \times n$ matrices A, as claimed.

42. a. We show first that D is linear in the ith row.

$$\underset{\uparrow \atop A}{D\begin{bmatrix} \vec{v}_1 \\ \vdots \\ \vec{x} \\ \vdots \\ \vec{v}_n \end{bmatrix}} = \frac{1}{\det M} \det \underset{\uparrow \atop AM}{\begin{bmatrix} \vec{v}_1 M \\ \vdots \\ \vec{x} M \\ \vdots \\ \vec{v}_n M \end{bmatrix}}$$

The entries in the ith row of AM are linear combinations of the components x_i of the vector $\vec{x}$, while the other entries of AM are constants. Therefore, $\det(AM)$ is a linear combination of the x_i (think about the patterns). Since $\frac{1}{\det M}$ is a constant, we have $D\begin{bmatrix} \vec{v}_1 \\ \vdots \\ \vec{x} \\ \vdots \\ \vec{v}_n \end{bmatrix} = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$

for some constants c_i, as claimed.

b. Secondly, we need to show that $D(B) = -D(A)$ if B is obtained from A by a row swap:

$$A = \begin{bmatrix} \vec{v}_1 \\ \vdots \\ \vec{v}_i \\ \vdots \\ \vec{v}_j \\ \vdots \\ \vec{v}_n \end{bmatrix} \longrightarrow B = \begin{bmatrix} \vec{v}_1 \\ \vdots \\ \vec{v}_j \\ \vdots \\ \vec{v}_i \\ \vdots \\ \vec{v}_n \end{bmatrix}$$

$$D(B) = \frac{1}{\det M} \det \begin{bmatrix} \vec{v}_1 M \\ \vdots \\ \vec{v}_j M \\ \vdots \\ \vec{v}_i M \\ \vdots \\ \vec{v}_n M \end{bmatrix} = -\frac{1}{\det M} \det \begin{bmatrix} \vec{v}_1 M \\ \vdots \\ \vec{v}_i M \\ \vdots \\ \vec{v}_j M \\ \vdots \\ \vec{v}_n M \end{bmatrix} = -D(A).$$

c. The property $D(I_n) = 1$ is obvious.

It now follows from Exercise 41 that $\det(A) = D(A) = \dfrac{\det(AM)}{\det(M)}$ and therefore $\det(AM) = \det(A)\det(M)$.

43. Note that matrix A_1 is invertible, since $\det(A_1) \neq 0$. Now

$$T\begin{bmatrix} \vec{y} \\ \vec{x} \end{bmatrix} = [A_1\ A_2]\begin{bmatrix} \vec{y} \\ \vec{x} \end{bmatrix} = A_1\vec{y} + A_2\vec{x} = \vec{0} \text{ when } A_1\vec{y} = -A_2\vec{x}, \text{ or,}$$

$\vec{y} = -A_1^{-1}A_2\vec{x}$. This shows that for every $\vec{x}$ there is a unique $\vec{y}$ (that is, $\vec{y}$ is a function of $\vec{x}$); furthermore, this function is linear, with matrix $M = -A_1^{-1}A_2$.

44. Using the approach of Exercise 43, we have $A_1 = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$, and

$M = -A_1^{-1}A_2 = \begin{bmatrix} 1 & -8 \\ -1 & 3 \end{bmatrix}$. The function is $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & -8 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Alternatively, we can solve the linear system

$$\begin{aligned} y_1 + 2y_2 + x_1 + 2x_2 &= 0 \\ 3y_1 + 7y_2 + 4x_1 + 3x_2 &= 0 \end{aligned}$$

Gaussian Elimination gives

$$\begin{aligned} y_1 - x_1 + 8x_2 &= 0 \\ y_2 + x_1 - 3x_2 &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} y_1 &= x_1 - 8x_2 \\ y_2 &= -x_1 + 3x_2 \end{aligned}$$

45. The standard matrix of T is $A = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 2 & 6 \\ 0 & 0 & 2 \end{bmatrix}$, so that $\det(T) = \det(A) = 8$.

46. The standard matrix of T is $A = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 3 & -12 \\ 0 & 0 & 9 \end{bmatrix}$, so that $\det(T) = \det(A) = 27$.

53. The standard matrix of T is $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$, so that $\det(T) = \det(A) = 16$.