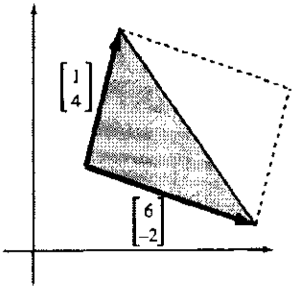
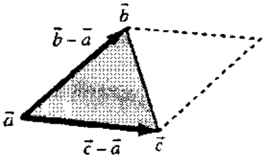
2. By Fact 6.3.3 Area $= \frac{1}{2}\left|\det\begin{bmatrix} 3 & 8 \\ 7 & 2 \end{bmatrix}\right| = \frac{1}{2}|-50| = 25$

3. Area of triangle $= \frac{1}{2}\left|\det\begin{bmatrix} 6 & 1 \\ -2 & 4 \end{bmatrix}\right| = 13$ (See figure.)



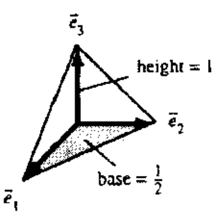
4. Note that area of triangle $= \dfrac{1}{2}\left|\det\begin{bmatrix} b_1 - a_1 & c_1 - a_1 \\ b_2 - a_2 & c_2 - a_2 \end{bmatrix}\right|$.



On the other hand,
$$\det\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{bmatrix} \underset{\substack{\uparrow \\ \text{subtract first column from} \\ \text{second and third}}}{=} \det\begin{bmatrix} a_1 & b_1 - a_1 & c_1 - a_1 \\ a_2 & b_2 - a_2 & c_2 - a_2 \\ 1 & 0 & 0 \end{bmatrix} \underset{\substack{\uparrow \\ \text{expand down the} \\ \text{first column}}}{=} \det\begin{bmatrix} b_1 - a_1 & c_1 - a_1 \\ b_2 - a_2 & c_2 - a_2 \end{bmatrix}.$$

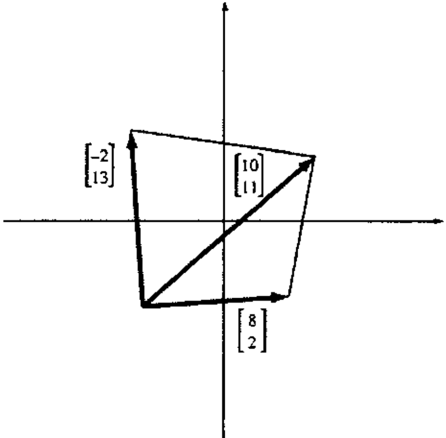
Therefore, area of triangle $= \dfrac{1}{2}\left|\det\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{bmatrix}\right|$.

5. The volume of the tetrahedron T_0 defined by $\vec{e}_1, \vec{e}_2, \vec{e}_3$ is $\frac{1}{3}(\text{base})(\text{height}) = \frac{1}{6}$ (formula for the volume of a pyramid).



The tetrahedron T defined by $\vec{v}_1, \vec{v}_2, \vec{v}_3$ can be obtained by applying the linear transformation with matrix $[\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$ to T_0.
Now we have
$$\text{vol}(T) \underset{\substack{\uparrow \\ \text{Fact 6.3.8 and page 277}}}{=} |\det[\vec{v}_1\ \vec{v}_2\ \vec{v}_3]|\text{vol}(T_0) = \tfrac{1}{6}|\det[\vec{v}_1\ \vec{v}_2\ \vec{v}_3]| \underset{\substack{\uparrow \\ \text{Fact 6.3.5}}}{=} \tfrac{1}{6}V(\vec{v}_1, \vec{v}_2, \vec{v}_3).$$

7. Area $= \frac{1}{2}\left|\det\begin{bmatrix}10 & -2\\ 11 & 13\end{bmatrix}\right| + \frac{1}{2}\left|\det\begin{bmatrix}8 & 10\\ 2 & 11\end{bmatrix}\right| = 110$ (See Figure.)

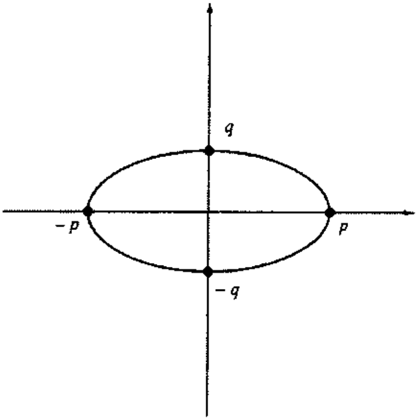


13. By Fact 6.3.7, the desired 2-volume is $\sqrt{\det\left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}\right)} = \sqrt{\det\begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix}} = \sqrt{20}.$

14. By Fact 6.3.7, the desired 3-volume is

$$\sqrt{\det\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 1 & 4 \end{bmatrix}\right)} = \sqrt{\det\begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 10 \\ 1 & 10 & 30 \end{bmatrix}} = \sqrt{6}.$$

18. a.

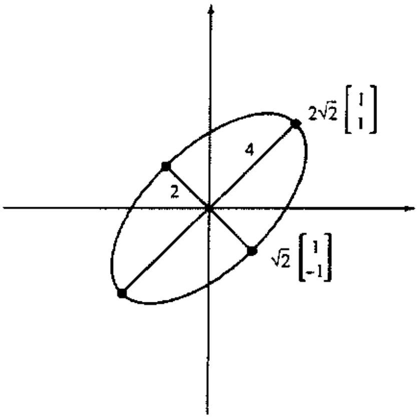


$\begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix} \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} = \begin{bmatrix} p \cdot \cos(t) \\ q \cdot \sin(t) \end{bmatrix}$, the ellipse with semi-axes $\pm \begin{bmatrix} p \\ 0 \end{bmatrix}$ and $\pm \begin{bmatrix} 0 \\ q \end{bmatrix}$.

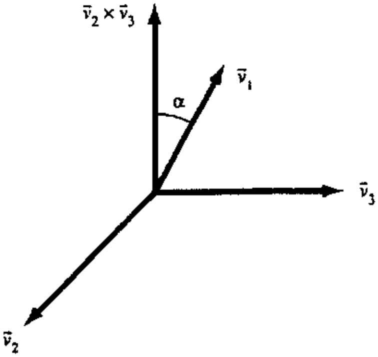
(area of the ellipse) $= |\det(A)|$(area of the unit circle) $= pq\pi$

b. By Fact 6.3.8, $|\det(A)| = \dfrac{\text{area of the ellipse}}{\text{area of the unit circle}} = \dfrac{ab\pi}{\pi} = ab$ so $|\det(A)| = ab$.

c. The unit circle consists of all vectors of the form $\vec{x} = \cos(t)\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \sin(t)\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$; its image is the ellipse consisting of all vectors $T(\vec{x}) = \cos(t) \underbrace{2\sqrt{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\text{semi-major axis}} + \sin(t) \underbrace{\sqrt{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\text{semi-minor axis}}$.



19. $\det[\,\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3\,] = \vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3) = \|\vec{v}_1\|\|\vec{v}_2 \times \vec{v}_3\| \cos\alpha$ where α is the angle between $\vec{v}_1$ and $\vec{v}_2 \times \vec{v}_3$ so $\det[\,\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3\,] > 0$ if and only if $\cos\alpha > 0$, i.e., if and only if α is acute $\left(0 \le \alpha < \frac{\pi}{2}\right)$.



20. By Exercise 19, $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ constitute a positively oriented basis if and only if $\det[\vec{v}_1\ \vec{v}_2\ \vec{v}_3] > 0$. Assume that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is such a basis. We want to show that $A\vec{v}_1, A\vec{v}_2, A\vec{v}_3$ is positively oriented if and only if $\det(A) > 0$. We have $\det[A\vec{v}_1\ A\vec{v}_2\ A\vec{v}_3] = \det(A[\vec{v}_1\ \vec{v}_2\ \vec{v}_3]) = \det(A)\det[\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$ so since $\det[\vec{v}_1\ \vec{v}_2\ \vec{v}_3] > 0$ by assumption, $\det[A\vec{v}_1\ A\vec{v}_2\ A\vec{v}_3] > 0$ if and only if $\det(A) > 0$. Hence A is orientation preserving if and only if $\det(A) > 0$.

21. a. Reverses

Consider $\vec{v}_2$ and $\vec{v}_3$ in the plane (not parallel), and let $\vec{v}_1 = \vec{v}_2 \times \vec{v}_3$; then $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is a positively oriented basis, but $T(\vec{v}_1) = -\vec{v}_1, T(\vec{v}_2) = \vec{v}_2, T(\vec{v}_3) = \vec{v}_3$ is negatively oriented.

b. Preserves

Consider $\vec{v}_2$ and $\vec{v}_3$ orthogonal to the line (not parallel), and let $\vec{v}_1 = \vec{v}_2 \times \vec{v}_3$; then $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is a positively oriented basis, and $T(\vec{v}_1) = \vec{v}_1, T(\vec{v}_2) = -\vec{v}_2, T(\vec{v}_3) = -\vec{v}_3$ is positively oriented as well.

c. Reverses

The standard basis $\vec{e}_1, \vec{e}_2, \vec{e}_3$ is positively oriented, but $T(\vec{e}_1) = -\vec{e}_1, T(\vec{e}_2) = -\vec{e}_2, T(\vec{e}_3) = -\vec{e}_3$ is negatively oriented.

22. Here $A = \begin{bmatrix} 3 & 7 \\ 4 & 11 \end{bmatrix}$, $\det(A) = 5$, $\vec{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, so by Fact 6.3.9

$$x = \frac{\det\begin{bmatrix} 1 & 7 \\ 3 & 11 \end{bmatrix}}{5} = -2,\ y = \frac{\det\begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix}}{5} = 1.$$

23. Here $A = \begin{bmatrix} 5 & -3 \\ -6 & 7 \end{bmatrix}$, $\det(A) = 17$, $\vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so by Fact 6.3.9

$$x_1 = \frac{\det\begin{bmatrix} 1 & -3 \\ 0 & 7 \end{bmatrix}}{17} = \frac{7}{17},\ x_2 = \frac{\det\begin{bmatrix} 5 & 1 \\ -6 & 0 \end{bmatrix}}{17} = \frac{6}{17}.$$

24. Here $A = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 4 & 5 \\ 6 & 0 & 7 \end{bmatrix}$, $\det(A) = 146$, $\vec{b} = \begin{bmatrix} 8 \\ 3 \\ -1 \end{bmatrix}$, so by Fact 6.3.9,