4. $\det(\lambda I_2 - A) = \det\begin{bmatrix} \lambda & -4 \\ 1 & \lambda - 4 \end{bmatrix} = \lambda(\lambda - 4) + 4 = (\lambda - 2)^2 = 0$ so $\lambda = 2$ with algebraic multiplicity 2.

5. $\det(\lambda I_2 - A) = \det\begin{bmatrix} \lambda - 11 & 15 \\ -6 & \lambda + 7 \end{bmatrix} = \lambda^2 - 4\lambda + 13$ so $\det(\lambda I_2 - A) = 0$ for no real λ.

6. $\det(\lambda I_2 - A) = \det\begin{bmatrix} \lambda - 1 & -2 \\ -3 & \lambda - 4 \end{bmatrix} = \lambda^2 - 5\lambda - 2 = 0$ so $\lambda_{1,2} = \dfrac{5 \pm \sqrt{33}}{2}$.

7. $\lambda = 1$ with algebraic multiplicity 3, by Fact 7.2.2.

8. $f_A(\lambda) = \lambda^2(\lambda + 3)$ so
$\lambda_1 = 0$ (Algebraic multiplicity 2)
$\lambda_2 = -3$.

9. $f_A(\lambda) = (\lambda - 2)^2(\lambda - 1)$ so
$\lambda_1 = 2$ (Algebraic multiplicity 2)
$\lambda_2 = 1$.

10. $f_A(\lambda) = (\lambda + 1)^2(\lambda - 1)$ so $\lambda_1 = -1$ (Algebraic multiplicity 2), $\lambda_2 = 1$.

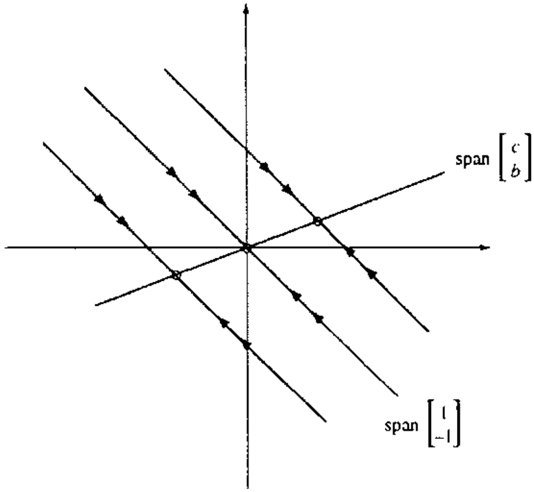
11. $f_A(\lambda) = \lambda^3 + \lambda^2 + \lambda + 1 = (\lambda + 1)(\lambda^2 + 1) = 0$
$\lambda = -1$ (Algebraic multiplicity 1).

12. $f_A(\lambda) = \lambda(\lambda + 1)(\lambda - 1)^2$ so $\lambda_1 = 0$, $\lambda_2 = -1$, $\lambda_3 = 1$ (Algebraic multiplicity 2).

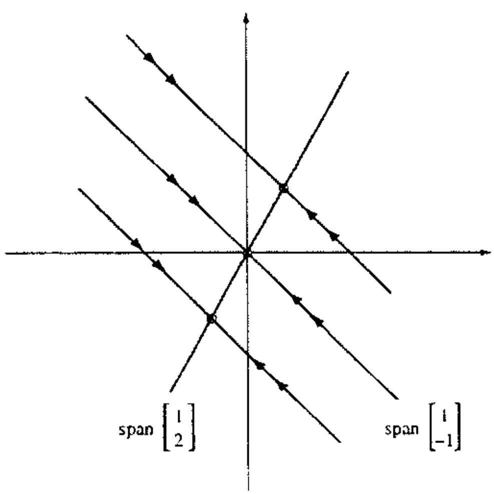
24. $\lambda_1 - 0.25,\ \lambda_2 = 1$

25. $A\begin{bmatrix} c \\ b \end{bmatrix} = \begin{bmatrix} ac + bc \\ bc + bd \end{bmatrix} = \begin{bmatrix} (a+b)c \\ (c+d)b \end{bmatrix} = \begin{bmatrix} c \\ b \end{bmatrix}$ since $a + b = c + d = 1$; therefore, $\begin{bmatrix} c \\ b \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda_1 = 1$.

Also, $A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a - c \\ b - d \end{bmatrix} = (a - c)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ since $a - c = -(b - d)$; therefore, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda_2 = a - c$. Notice that $|a - c| < 1$ so a possible phase portrait is



26. Here $\begin{bmatrix} c \\ b \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.5 \end{bmatrix}$ with $\lambda_1 = 1$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ with $\lambda_2 = a - c = 0.25$.



27. a. We know $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \lambda_1 = 1$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \lambda_2 = \frac{1}{4}$. If $\vec{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ then $\vec{x}_0 = \frac{1}{3}\vec{v}_1 + \frac{2}{3}\vec{v}_2$, so by Fact 7.1.3,

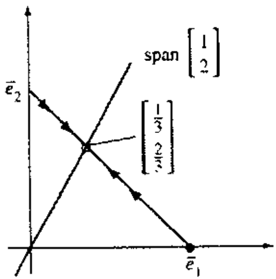
$$x_1(t) = \frac{1}{3} + \frac{2}{3}\left(\frac{1}{4}\right)^t$$

$$x_2(t) = \frac{2}{3} - \frac{2}{3}\left(\frac{1}{4}\right)^t.$$

If $\vec{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then $\vec{x}_0 = \frac{1}{3}\vec{v}_1 - \frac{1}{3}\vec{v}_2$, so by Fact 7.1.3,

$$x_1(t) = \frac{1}{3} - \frac{1}{3}\left(\frac{1}{4}\right)^t$$

$$x_2(t) = \frac{2}{3} + \frac{1}{3}\left(\frac{1}{4}\right)^t$$



b. A^t approaches $\frac{1}{3}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, as $t \to \infty$. See part c for a justification.

c. Let us think about the first column of A^t, which is $A^t\vec{e}_1$. We can use Fact 7.1.3 to compute $A^t\vec{e}_1$.

Start by writing $\vec{e}_1 = c_1\begin{bmatrix} c \\ b \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}$; a straightforward computation shows that $c_1 = \dfrac{1}{c+b}$ and $c_2 = \dfrac{b}{c+b}$.

Now $A^t\vec{e}_1 = \dfrac{1}{c+b}\begin{bmatrix} c \\ b \end{bmatrix} + \dfrac{b}{c+b}(\lambda_2)^t\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, where $\lambda_2 = a - c$.

Since $|\lambda_2| < 1$, the second summand goes to zero, so that $\lim\limits_{t\to\infty}(A^t\vec{e}_1) = \dfrac{1}{c+b}\begin{bmatrix} c \\ b \end{bmatrix}$.

Likewise, $\lim\limits_{t\to\infty}(A^t\vec{e}_2) = \dfrac{1}{c+b}\begin{bmatrix} c \\ b \end{bmatrix}$, so that $\lim\limits_{t\to\infty} A^t = \dfrac{1}{c+b}\begin{bmatrix} c & c \\ b & b \end{bmatrix}$.