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10. Fails to be diagonalizable. There is only one eigenvalue, 1, with a one-dimensional eigenspace.
11. Fails to be diagonalizable. The eigenvalues are 1,2,1, and the eigenspace $E_1 = \ker(I_3 - A) = \text{span}(\vec{e}_1)$ is only one-dimensional.

12. Diagonalizable. The eigenvalues are 2,1,1, with associated eigenvectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. If we let

$$S = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ then } S^{-1}AS = D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

15. Diagonalizable. The eigenvalues are 1, -1, 1, with associated eigenvectors $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. If we let

$$S = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ then } S^{-1}AS = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

16. Diagonalizable. The eigenvalues are 3,2,1, with associated eigenvectors $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. If we let

$$S = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \text{ then } S^{-1}AS = D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

17. Diagonalizable. The eigenvalues are 0,3,0, with associated eigenvectors $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. If we let

$$S = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \text{ then } S^{-1}AS = D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

18. Diagonalizable. The eigenvalues are 3,2,1, with associated eigenvectors $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. If we let

$$S = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \text{ then } S^{-1}AS = D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

19. Fails to be diagonalizable. The eigenvalues are 1,0,1, and the eigenspace $E_1 = \ker(I_3 - A) = \text{span}(\vec{e}_1)$ is only one-dimensional.

32. The eigenvalues of $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$ are 3 and 2, with associated eigenvectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. If we let

$S = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, then $S^{-1}AS = D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$. Thus

$$A = SDS^{-1} \text{ and } A^t = SD^tS^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3^t & 0 \\ 0 & 2^t \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2(3^t) - 2^t & 2^{t+1} - 2(3^t) \\ 3^t - 2^t & 2^{t+1} - 3^t \end{bmatrix}$$

48. Apply T to the standard basis: $T(1) = 1$, $T(x) = 2x$, and $T(x^2) = (2x)^2 = 4x^2$. This gives the eigenvalues 1, 2, and 4, with corresponding eigenfunctions 1, x , x^2 . Yes, T is diagonalizable, since the standard basis is an eigenbasis for T .

54. Note that $A^2 = 0$, but $B^2 \neq 0$. Since A^2 fails to be similar to B^2 , matrix A isn't similar to B (see Example 7 of Section 3.4).

59. If \vec{v} is an eigenvector with eigenvalue λ , then

$$\begin{aligned}f_A(A)\vec{v} &= (A^n + a_{n-1}A^{n-1} + \cdots + a_1A + a_0I_n)\vec{v} \\&= \lambda^n\vec{v} + a_{n-1}\lambda^{n-1}\vec{v} + \cdots + a_1\lambda\vec{v} + a_0\vec{v} \\&= (\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0)\vec{v} \\&= f_A(\lambda)\vec{v} = 0\vec{v} = \vec{0}.\end{aligned}$$

Since A is diagonalizable, any vector \vec{x} in \mathbb{R}^n can be written as a linear combination of eigenvectors, so that $f_A(A)\vec{x} = \vec{0}$. Since this equation holds for all \vec{x} in \mathbb{R}^n , we have $f_A(A) = 0$, as claimed.

61. a. B is diagonalizable since it has three distinct eigenvalues, so that $S^{-1}BS$ is diagonal for some invertible S . But $S^{-1}AS = S^{-1}I_3S = I_3$ is diagonal as well. Thus A and B are indeed simultaneously diagonalizable.
- b. There is an invertible S such that $S^{-1}AS = D_1$ and $S^{-1}BS = D_2$ are both diagonal. Then $A = SD_1S^{-1}$ and $B = SD_2S^{-1}$, so that $AB = (SD_1S^{-1})(SD_2S^{-1}) = SD_1D_2S^{-1}$ and $BA = (SD_2S^{-1})(SD_1S^{-1}) = SD_2D_1S^{-1}$. These two results agree, since $D_1D_2 = D_2D_1$ for the diagonal matrices D_1 and D_2 .
- c. Let A be I_n and B a nondiagonalizable $n \times n$ matrix, for example, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.
- d. Suppose $BD = DB$ for a diagonal D with distinct diagonal entries. The ij th entry of the matrix $BD = DB$ is $b_{ij}d_{jj} = d_{ii}b_{ij}$. For $i \neq j$ this implies that $b_{ij} = 0$. Thus B must be diagonal.
- e. Since A has n distinct eigenvalues, A is diagonalizable, that is, there is an invertible S such that $S^{-1}AS = D$ is a diagonal matrix with n distinct diagonal entries. We claim that $S^{-1}BS$ is diagonal as well; by part d it suffices to show that $S^{-1}BS$ commutes with $D = S^{-1}AS$. This is easy to verify: $(S^{-1}BS)D = (S^{-1}BS)(S^{-1}AS) = S^{-1}BAS = S^{-1}ABS = (S^{-1}AS)(S^{-1}BS) = D(S^{-1}BS)$.