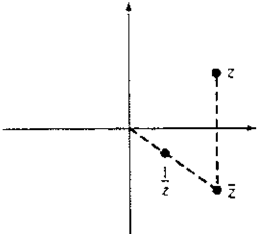
4. Let $z = r(\cos\phi + i\sin\phi)$ then $w = \sqrt{r}\left(\cos\left(\frac{\phi + 2\pi k}{2}\right) + i\sin\left(\frac{\phi + 2\pi k}{2}\right)\right)$, $k = 0,\ 1$.

5. Let $z = r(\cos\phi + i\sin\phi)$ then $w = \sqrt[n]{r}\left(\cos\left(\frac{\phi + 2\pi k}{n}\right) + i\sin\left(\frac{\phi + 2\pi k}{n}\right)\right)$, $k = 0, 1, 2, \ldots, n-1$.

6. If we have $z = r(\cos\phi + i\sin\phi)$ then $\frac{1}{z}$ must have the property that $z \cdot \frac{1}{z} = 1 = \cos 0 + i\sin 0$ i.e. $|z| \cdot \left|\frac{1}{z}\right| = 1$ and $\arg\left(z \cdot \frac{1}{z}\right) = \arg(z) + \arg\left(\frac{1}{z}\right) = 0$ so $\frac{1}{z} = \frac{1}{r}(\cos(-\phi) + i\sin(-\phi)) = \frac{1}{r}(\cos\phi - i\sin\phi)$ (since cosine is even, sine odd). Hence $\frac{1}{z}$ is a real scalar multiple of z.


z
1/z
z̄

12. We will use the facts:

i) $\overline{z+w} = \overline{z} + \overline{w}$ and

ii) $(\overline{z^n}) = \overline{z}^n$

These are easy to check. Assume λ_0 is a complex root of $f(\lambda) = a_n\lambda^n + \cdots + a_1\lambda + a_0$ where the coefficients a_i are real. Since λ_0 is a root of f, we have $a_n\lambda_0^n + a_{n-1}\lambda_0^{n-1} + \cdots + a_1\lambda_0 + a_0 = 0$.

Taking the conjugate of both sides we get $\overline{a_n\lambda_0^n + a_{n-1}\lambda_0^{n-1} + \cdots + a_1\lambda_0 + a_0} = \bar{0}$ so by fact i), and factoring the real constants we get $a_n\overline{\lambda_0^n} + a_{n-1}\overline{\lambda_0^{n-1}} + \cdots + a_1\overline{\lambda_0} + a_0 = 0$.

Now, by fact ii), $a_n(\overline{\lambda}_0)^n + a_{n-1}(\overline{\lambda}_0)^{n-1} + \cdots + a_1\overline{\lambda}_0 + a_0 = 0$, i.e. $\overline{\lambda}_0$ is also a root of f, as claimed.

22. $f_A(\lambda) = (\lambda - 1)(\lambda - 10) + 12 = \lambda^2 - 11\lambda + 22$ so $\lambda_{1,2} = \dfrac{11 \pm \sqrt{33}}{2}$.

23. $f_A(\lambda) = \lambda^3 + 1 = (\lambda - 1)(\lambda^2 + \lambda + 1)$ so $\lambda_1 = 1, \lambda_{2,3} = \dfrac{-1 \pm \sqrt{3}i}{2}$.

24. $f_A(\lambda) = \lambda^3 - 3\lambda^2 + 7\lambda - 5$ so $\lambda_1 = 1, \lambda_{2,3} = 1 \pm 2i$. (See Exercise 11.)

25. $f_A(\lambda) = \lambda^4 - 1 = (\lambda^2 - 1)(\lambda^2 + 1) = (\lambda - 1)(\lambda + 1)(\lambda - i)(\lambda + i)$ so $\lambda_{1,2} = \pm 1$ and $\lambda_{3,4} = \pm i$

26. $f_A(\lambda) = (\lambda^2 - 2\lambda + 2)(\lambda^2 - 2\lambda) = (\lambda^2 - 2\lambda + 2)(\lambda - 2)\lambda = 0$, so $\lambda_{1,2} = 1 \pm i, \lambda_3 = 2, \lambda_4 = 0$.

27. By Fact 7.5.5 $\text{tr}(A) = \lambda_1 + \lambda_2 + \lambda_3$, $\det(A) = \lambda_1\lambda_2\lambda_3$ but $\lambda_1 = \lambda_2 \neq \lambda_3$ by assumption, so $\text{tr}(A) = 1 = 2\lambda_2 + \lambda_3$ and $\det(A) = 3 = \lambda_2^2\lambda_3$.
Solving for λ_2, λ_3 we get $-1, 3$ hence $\lambda_1 = \lambda_2 = -1$ and $\lambda_3 = 3$. (Note that the eigenvalues must be real; why?)

28. By Fact 7.5.5
$2z\bar{z} = 50$
$2 + z + \bar{z} = 8$
If $z = a + bi$ we get $(a + bi) + (a - bi) = 6$ so $a = 3$.
$(a + bi)(a - bi) = 25$, i.e. $a^2 + b^2 = 25$ or $9 + b^2 = 25$ so $b = \pm 4$.
Hence $\lambda_{2,3} = 3 \pm 4i$.

29. $\text{tr}(A) = 0$ so $\lambda_1 + \lambda_2 + \lambda_3 = 0$.
Also, we can compute $\det(A) = bcd > 0$ since $b, c, d > 0$. Therefore, $\lambda_1\lambda_2\lambda_3 > 0$.
Hence two of the eigenvalues must be negative, and the largest one (in absolute value) must be positive.

30. a. The ith entry of $A\vec{x}$ is $\sum_{k=1}^{n} a_{ik}x_k$, so that the sum of all the entries of $A\vec{x}$ is

$$\sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}x_k = \sum_{k=1}^{n}\sum_{i=1}^{n} a_{ik}x_k = \sum_{k=1}^{n}\Big(\underbrace{\sum_{i=1}^{n} a_{ik}}_{1}\Big)x_k = \sum_{k=1}^{n} x_k = 1.$$

b. As we do some computer experiments, A^t appears to approach a matrix with identical columns, with column sum 1. Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be an eigenbasis with $\lambda_1 = 1$ and $|\lambda_j| < 1$ for $j = 2, \ldots, n$. For a *fixed* i, write $\vec{e}_i = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$, so that ($i$th column of A^t) $= A^t\vec{e}_i = c_1\vec{v}_1 + [c_2\lambda_2^t\vec{v}_2 + \cdots + c_n\lambda_n^t\vec{v}_n]$.
(The term in *square* brackets goes to zero as t goes to infinity.)
Therefore, $\lim_{t\to\infty}$ (ith column of A^t) $= \lim_{t\to\infty}(A^t\vec{e}_i) = c_1\vec{v}_1$.
Furthermore, the entries of $A^t\vec{e}_i$ add up to 1, for all t, by part a. Therefore, the same is true for the limit (since the limit of a sum is the sum of the limits).
It follows that $\lim_{t\to\infty}(A^t)$ exists and has identical columns, with column sum 1, as claimed.

31. No matter how we choose A, $\frac{1}{15}A$ is a regular transition matrix, so that $\lim_{t\to\infty}\left(\frac{1}{15}A\right)^t$ is a matrix with identical columns by Exercise 30. Therefore, the columns of A^t "become more and more alike" as t approaches infinity, in the sense that $\lim_{t\to\infty}\dfrac{ij\text{th entry of } A^t}{ik\text{th entry of } A^t} = 1$ for all i, j, k.

32. a. $\vec{x}(t) = \begin{bmatrix} a(t) \\ m(t) \\ s(t) \end{bmatrix} = \begin{bmatrix} 0.6a(t) + 0.1m(t) + 0.5s(t) \\ 0.2a(t) + 0.7m(t) + 0.1s(t) \\ 0.2a(t) + 0.2m(t) + 0.4s(t) \end{bmatrix}$ so $A = \begin{bmatrix} 0.6 & 0.1 & 0.5 \\ 0.2 & 0.7 & 0.1 \\ 0.2 & 0.2 & 0.4 \end{bmatrix}$.

Note that A is a regular transition matrix.

b. By Exercise 30, $\lim_{t\to\infty}(A^t) = [\vec{v}\,\vec{v}\,\vec{v}]$, where $\vec{v}$ is the unique eigenvector of A with eigenvalue 1 and column sum 1. We find that $\vec{v} = \begin{bmatrix} 0.4 \\ 0.35 \\ 0.25 \end{bmatrix}$.

Now $\lim_{t\to\infty} \vec{x}(t) = \lim_{t\to\infty}(A^t\vec{x}_0) = \left(\lim_{t\to\infty} A^t\right)\vec{x}_0 = [\vec{v}\,\vec{v}\,\vec{v}]\vec{x}_0 = \vec{v}$, since the components of $\vec{x}_0$ add up to 1. The market shares approach 40%, 35%, and 25%, respectively, regardless of the initial shares.