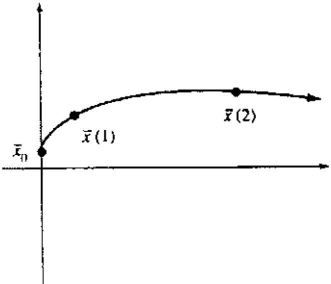
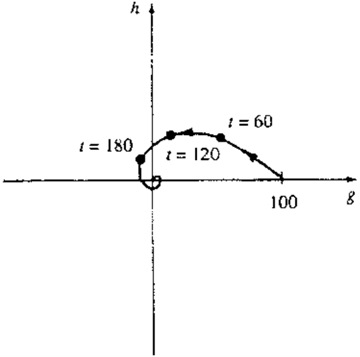
20. $\lambda_{1,2} = 4 \pm 3i, r = 5, \phi = \arctan\left(\frac{3}{4}\right) \approx 0.64$, so $\lambda_1 \approx 5(\cos(0.64) + i\sin(0.64))$, $[\vec{w}\ \vec{v}] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{x}(t) \approx 5^t \begin{bmatrix} \sin(0.64t) \\ \cos(0.64t) \end{bmatrix}$.

Spirals outwards (rotation-dilation).



32. $\lambda_{1,2} = 0.99 \pm (0.01)i$ with eigenvector $\begin{bmatrix} 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for λ_1, so $\vec{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\phi = \arctan\left(\frac{0.01}{0.99}\right) \approx 0.01$, $r \approx 0.99$. Hence $\vec{x}(0) = \begin{bmatrix} 100 \\ 0 \end{bmatrix} = 100\vec{w} + 0\vec{v}$ and

$\vec{x}(t) \approx 0.99^t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(0.01t) & -\sin(0.01t) \\ \sin(0.01t) & \cos(0.01t) \end{bmatrix} \begin{bmatrix} 100 \\ 0 \end{bmatrix} = 0.99^t \cdot 100 \begin{bmatrix} \cos(0.01t) \\ \sin(0.01t) \end{bmatrix}$ i.e. $\vec{x}(t)$ spirals in.



Hence, both glucose and excess insulin oscillate with damped oscillations until all of the excess sugar has been eliminated.

38. a. $T(\vec{v}) = A\vec{v} + \vec{b} = \vec{v}$ if $\vec{v} - A\vec{v} = \vec{b}$ or $(I_n - A)\vec{v} = \vec{b}$.
$I_n - A$ is invertible since 1 is not an eigenvalue of A. Therefore, $\vec{v} = (I_n - A)^{-1}\vec{b}$ is the only solution.

b. Let $\vec{y}(t) = \vec{x}(t) - \vec{v}$ be the deviation of $\vec{x}(t)$ from the equilibrium $\vec{v}$.
Then $\vec{y}(t+1) = \vec{x}(t+1) - \vec{v} = A\vec{x}(t) + \vec{b} - \vec{v} = A(\vec{y}(t) + \vec{v}) + \vec{b} - \vec{v} = A\vec{y}(t) + A\vec{v} + \vec{b} - \vec{v} = A\vec{y}(t)$,
so that $\vec{y}(t) = A^t\vec{y}(0)$, or $\vec{x}(t) = \vec{v} + A^t(\vec{x}_0 - \vec{v})$.
$\lim_{t\to\infty} \vec{x}(t) = \vec{v}$ for all $\vec{x}_0$ if $\lim_{t\to\infty} A^t(\vec{x}_0 - \vec{v}) = \vec{0}$. This is the case if the modulus of all the eigenvalues of A is less than 1.

40. Note that A can be partitioned as $A = \begin{bmatrix} B & -C^T \\ C & B^T \end{bmatrix}$, where B and C are rotation-dilation matrices. Also note that $BC = CB, B^TB = (p^2+q^2)I_2$, and $C^TC = (r^2+s^2)I_2$.

a. $A^TA = \begin{bmatrix} B^T & C^T \\ -C & B \end{bmatrix}\begin{bmatrix} B & -C^T \\ C & B^T \end{bmatrix} = (p^2+q^2+r^2+s^2)I_4$

b. By part a, $A^{-1} = \dfrac{1}{p^2+q^2+r^2+s^2}A^T$ if $A \neq 0$.

c. $(\det A)^2 = (p^2+q^2+r^2+s^2)^4$, by part a, so that $\det A = \pm(p^2+q^2+r^2+s^2)^2$.
The diagonal pattern makes the contribution $+p^4$, so that $\det A = (p^2+q^2+r^2+s^2)^2$.

d. Consider $\det(\lambda I_4 - A)$. Note that the matrix $\lambda I_4 - A$ has the same "format" as A, with p replaced by $\lambda - p$ and q, r, s by $-q, -r, -s$, respectively. By part c, $\det(\lambda I_4 - A) = ((\lambda-p)^2+q^2+r^2+s^2)^2 = 0$ when

$(\lambda-p)^2 = -q^2-r^2-s^2$

$\lambda - p = \pm i\sqrt{q^2+r^2+s^2}$

$\lambda_{1,2} = p \pm i\sqrt{q^2+r^2+s^2}$

Each of these eigenvalues has algebraic multiplicity 2 (if $q = r = s = 0$ then $\lambda = p$ has algebraic multiplicity 4).

e. By part a we can write $A = \sqrt{p^2+q^2+r^2+s^2}\underbrace{\left(\dfrac{1}{\sqrt{p^2+q^2+r^2+s^2}}A\right)}_{S}$, where S is orthogonal.

Therefore, $\|A\vec{x}\| = \|\sqrt{p^2+q^2+r^2+s^2}(S\vec{x})\| = \sqrt{p^2+q^2+r^2+s^2}\|\vec{x}\|$.

f. Let $A = \begin{bmatrix} 3 & -3 & -4 & -5 \\ 3 & 3 & 5 & -4 \\ 4 & -5 & 3 & 3 \\ 5 & 4 & -3 & 3 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 4 \end{bmatrix}$; then $A\vec{x} = \begin{bmatrix} -39 \\ 13 \\ 18 \\ 13 \end{bmatrix}$.

By part e, $\|A\vec{x}\|^2 = (3^2+3^2+4^2+5^2)\|\vec{x}\|^2$, or
$39^2+13^2+18^2+13^2 = (3^2+3^2+4^2+5^2)(1^2+2^2+4^2+4^2)$, as desired.

g. Any positive integer m can be written as $m = p_1p_2\ldots p_n$. Using part f repeatedly we see that the numbers $p_1, p_1p_2, p_1p_2p_3, \ldots, p_1p_2p_3\cdots p_{n-1}$, and finally $m = p_1\cdots p_n$ can be expressed as the sums of four squares.