6. $\frac{1}{3}\begin{bmatrix}2\\2\\1\end{bmatrix}, \frac{1}{3}\begin{bmatrix}2\\-1\\-2\end{bmatrix}, \frac{1}{3}\begin{bmatrix}1\\-2\\2\end{bmatrix}$ is an orthonormal eigenbasis.

10. $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 9$.

$\vec{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ is in E_0 and $\vec{v}_2 = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ is in E_9.

Let $\vec{v}_3 = \vec{v}_1 \times \vec{v}_2 = \frac{1}{3\sqrt{5}}\begin{bmatrix} 2 \\ -4 \\ -5 \end{bmatrix}$; then $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is an orthonormal eigenbasis.

$$S = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{3} & \frac{2}{3\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{3} & -\frac{4}{3\sqrt{5}} \\ 0 & \frac{2}{3} & -\frac{\sqrt{5}}{3} \end{bmatrix} \text{ and } D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

11. $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is an orthonormal eigenbasis, with $\lambda_1 = 2$, $\lambda_2 = 0$, and $\lambda_3 = 1$, so

$$S = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & 1 & 0 \end{bmatrix} \text{ and } D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

12. a. $E_1 = \text{span}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $E_{-1} = (E_1)^{\perp}$. An orthonormal eigenbasis is $\frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$.

b. Use Fact 7.4.1: $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

c. $A = SBS^{-1} = \begin{bmatrix} -0.6 & 0 & 0.8 \\ 0 & -1 & 0 \\ 0.8 & 0 & 0.6 \end{bmatrix}$, where $S = \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \\ \frac{2}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \end{bmatrix}$.

16. a. $\ker(A)$ is four-dimensional, so that the eigenvalue 0 has multiplicity 4, and the remaining eigenvalue is $\mathrm{tr}(A) = 5$.

b. $B = A + 2I_5$, so that the eigenvalues are 2, 2, 2, 2, 7.

c. $\det(B) = 2^4 \cdot 7 = 112$ (product of eigenvalues)

19. Let $L(\vec{x}) = A\vec{x}$. Then A^TA is symmetric, since $(A^TA)^T = A^T(A^T)^T = A^TA$, so that there is an orthonormal eigenbasis $\vec{v}_1, \ldots, \vec{v}_n$ for A^TA. Then the vectors $A\vec{v}_1, \ldots, A\vec{v}_n$ are orthogonal, since $A\vec{v}_i \cdot A\vec{v}_j = (A\vec{v}_i)^T A\vec{v}_j = \vec{v}_i^T A^T A\vec{v}_j = \vec{v}_i \cdot (A^TA\vec{v}_j) = \vec{v}_i \cdot (\lambda_j \vec{v}_j) = \lambda_j(\vec{v}_i \cdot \vec{v}_j) = 0$ if $i \neq j$.

20. By Exercise 19, there is an orthonormal basis $\vec{v}_1, \ldots, \vec{v}_n$ of $\mathbb{R}^n$ such that $T(\vec{v}_1), \ldots, T(\vec{v}_n)$ are orthogonal. Suppose that $T(\vec{v}_1), \ldots, T(\vec{v}_r)$ are nonzero and $T(\vec{v}_{r+1}), \ldots, T(\vec{v}_n)$ are zero. Then let $\vec{w}_i = \frac{1}{\|T(\vec{v}_i)\|} T(\vec{v}_i)$ for $i = 1, \ldots, r$ and choose an orthonormal basis $\vec{w}_{r+1}, \ldots, \vec{w}_m$ of $[\text{span}(\vec{w}_1, \ldots, \vec{w}_r)]^\perp$. Then $\vec{w}_1, \ldots, \vec{w}_m$ does the job.