6. $A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$, indefinite (since $\det(A) < 0$)

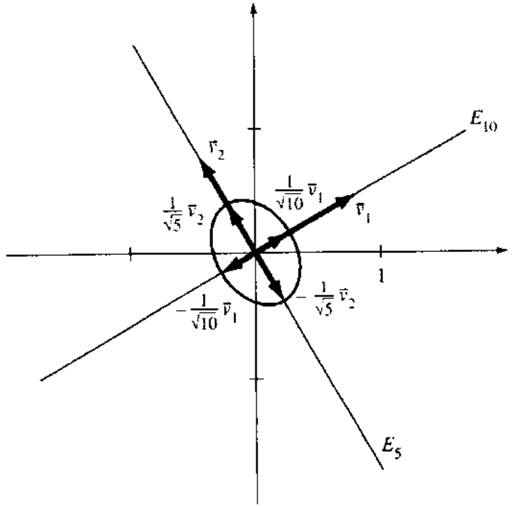
7. $A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 0 \end{bmatrix}$, indefinite (eigenvalues 2, -2, 3)

8. If $S^{-1}AS = D$ is diagonal, then $S^{-1}A^2S = D^2$, so that all eigenvalues of A^2 are ≥ 0. So A^2 is positive semi-definite; it is positive definite if and only if A is invertible.

18. $A = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}$, eigenvalues $\lambda_1 = 10$, $\lambda_2 = 5$

orthonormal eigenbasis $\vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

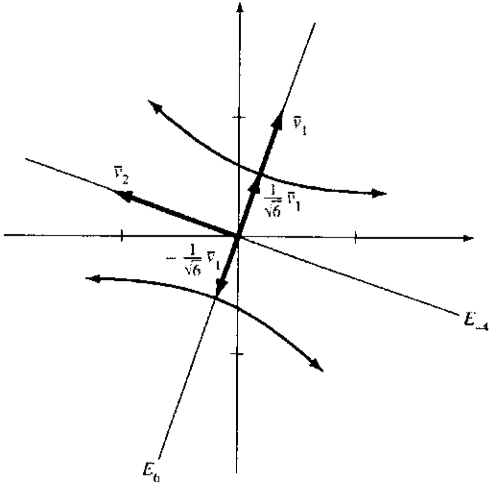
$10c_1^2 + 5c_2^2 = 1$ (ellipse)



20. $A = \begin{bmatrix} -3 & 3 \\ 3 & 5 \end{bmatrix}$; eigenvalues $\lambda_1 = 6$ and $\lambda_2 = -4$

orthonormal eigenbasis $\vec{v}_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\vec{v}_2 = \frac{1}{\sqrt{10}} \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

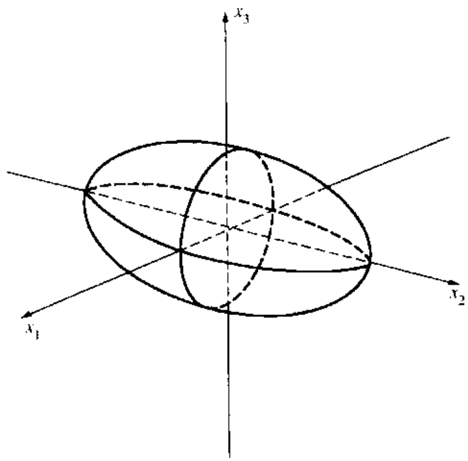
$6c_1^2 - 4c_2^2 = 1$ (hyperbola)



21. a. In each case, it is informative to think about the intersections with the three coordinate planes: $x_1 - x_2$, $x_1 - x_3$, and $x_2 - x_3$.

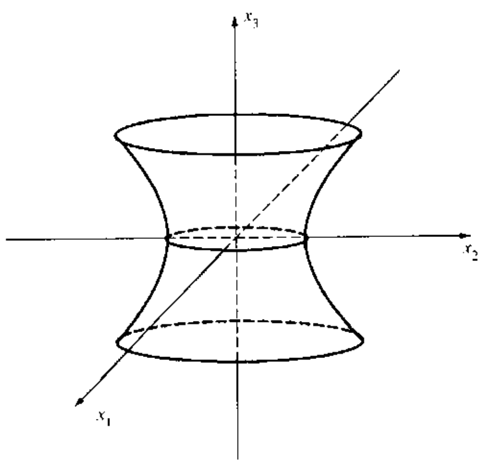
For the surface $x_1^2 + 4x_2^2 + 9x_3^2 = 1$, all these intersections are *ellipses*, and the surface itself is an *ellipsoid*.

This surface is connected and bounded; the points closest to the origin are $\pm \begin{bmatrix} 0 \\ 0 \\ \frac{1}{3} \end{bmatrix}$, and those farthest $\pm \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.



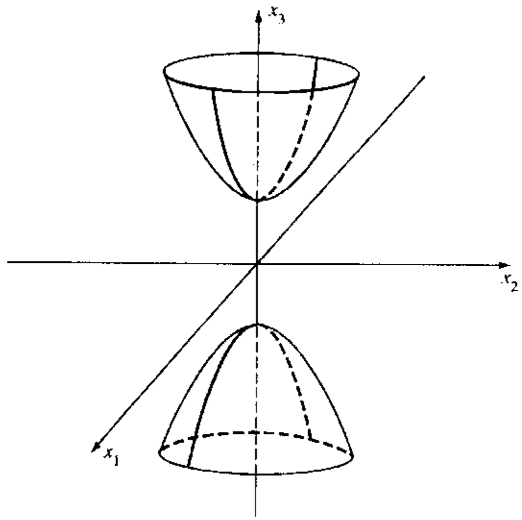
$x_1^2 + 4x_2^2 + 9x_3^2 = 1$ (not to scale)
an *ellipsoid*

In the case of $x_1^2 + 4x_2^2 - 9x_3^2 = 1$, the intersection with the $x_1 - x_2$ plane is an ellipse, and the two other intersections are hyperbolas. The surface is connected and not bounded; the points closest to the origin are $\pm \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix}$.



$x_1^2 + 4x_2^2 - 9x_3^2 = 1$ (not to scale)
a *hyperboloid of one sheet*

In the case $-x_1^2 - 4x_2^2 + 9x_3^2 = 1$, the intersection with the $x_1 - x_2$ plane is empty, and the two other intersections are hyperbolas. The surface consists of two pieces and is unbounded. The points closest to the origin are $\pm \begin{bmatrix} 0 \\ 0 \\ \frac{1}{3} \end{bmatrix}$.



$-x_1^2 - 4x_2^2 + 9x_3^2 = 1$ (not to scale)
a *hyperboloid of two sheets*

b. $A = \begin{bmatrix} 1 & \frac{1}{2} & 1 \\ \frac{1}{2} & 2 & \frac{3}{2} \\ 1 & \frac{3}{2} & 3 \end{bmatrix}$ is positive definite, with three positive eigenvalues $\lambda_1, \lambda_2, \lambda_3$.

Surface is given by $\lambda_1 c_1^2 + \lambda_2 c_2^2 + \lambda_3 c_3^2 = 1$ with respect to principal axes, an *ellipsoid*. To find points closest to and farthest from origin, use technology to find eigenvalues and eigenvectors:
eigenvalues: $\lambda_1 \approx 0.56, \lambda_2 \approx 4.44, \lambda_3 = 1$

unit eigenvectors: $\vec{v}_1 \approx \begin{bmatrix} 0.86 \\ 0.19 \\ -0.47 \end{bmatrix}, \vec{v}_2 \approx \begin{bmatrix} 0.31 \\ 0.54 \\ 0.78 \end{bmatrix}, \vec{v}_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

Equation: $0.56c_1^2 + 4.44c_2^2 + c_3^2 = 1$

Farthest points when $c_1 = \pm \frac{1}{\sqrt{0.56}}$ and $c_2 = c_3 = 0$

Closest points when $c_2 = \pm \frac{1}{\sqrt{4.44}}$ and $c_1 = c_3 = 0$

Farthest points $\approx \pm \frac{1}{\sqrt{0.56}} \begin{bmatrix} 0.86 \\ 0.19 \\ -0.47 \end{bmatrix} \approx \pm \begin{bmatrix} 1.15 \\ 0.26 \\ -0.63 \end{bmatrix}$

$$\text{Closest points} \approx \pm \frac{1}{\sqrt{4.44}} \begin{bmatrix} 0.31 \\ 0.54 \\ 0.78 \end{bmatrix} \approx \pm \begin{bmatrix} 0.15 \\ 0.26 \\ 0.37 \end{bmatrix}$$

22. $A = \begin{bmatrix} -1 & 0 & 5 \\ 0 & 1 & 0 \\ 5 & 0 & -1 \end{bmatrix}$; eigenvalues $\lambda_1 = 4$, $\lambda_2 = -6$, $\lambda_3 = 1$

Equation with respect to principal axes: $4c_1^2 - 6c_2^2 + c_3^2 = 1$, a hyperboloid of one sheet (see solutions to 21a).

Closest to origin when $c_1 = \pm\frac{1}{2}$, $c_2 = c_3 = 0$.

A unit eigenvector for eigenvalue 4 is $\vec{v} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, so that the desired points are $\pm\frac{1}{2}\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \approx$

$\pm \begin{bmatrix} 0.35 \\ 0 \\ 0.35 \end{bmatrix}$.